Study material of Class XII Science Stream

Chapter 5

Capacitance and Dielectrics

Capacitance and Dielectrics

5.1 Introduction

A capacitor is a device which stores electric charge. Capacitors vary in shape and size, but the basic configuration is two conductors carrying equal but opposite charges (Figure 5.1.1). Capacitors have many important applications in electronics. Some examples include storing electric potential energy, delaying voltage changes when coupled with resistors, filtering out unwanted frequency signals, forming resonant circuits and making frequency-dependent and independent voltage dividers when combined with resistors. Some of these applications will be discussed in latter chapters.

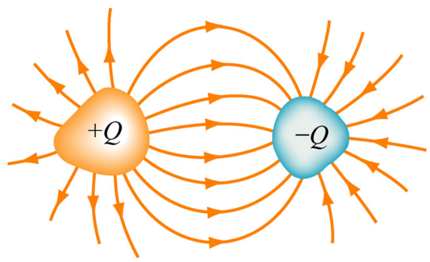


Figure 5.1.1 Basic configuration of a capacitor.

In the *uncharged* state, the charge on either one of the conductors in the capacitor is zero. During the charging process, a charge Q is moved from one conductor to the other one, giving one conductor a charge $+Q$, and the other one a charge $-Q$. A potential difference ΔV is created, with the positively charged conductor at a higher potential than the negatively charged conductor. Note that whether charged or uncharged, the net charge on the capacitor as a whole is zero.

The simplest example of a capacitor consists of two conducting plates of area A, which are parallel to each other, and separated by a distance *d*, as shown in Figure 5.1.2.

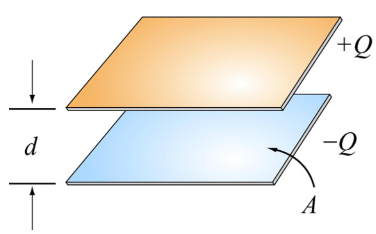


Figure 5.1.2 A parallel-plate capacitor

Experiments show that the amount of charge Q stored in a capacitor is linearly proportional to ΔV, the electric potential difference between the plates. Thus, we may write

$$Q = C\,|\Delta V| \tag{5.1.1}$$

where C is a positive proportionality constant called *capacitance*. Physically, capacitance is a measure of the capacity of storing electric charge for a given potential difference ΔV. The SI unit of capacitance is the *farad* (F):

$$1\ \text{F} = 1\ \text{farad} = 1\ \text{coulomb/volt} = 1\ \text{C/V}$$

A typical capacitance is in the picofarad ($1\ \text{pF} = 10^{-12}\,\text{F}$) to millifarad range, ($1\ \text{mF} = 10^{-3}\,\text{F} = 1000\,\mu\text{F};\ 1\,\mu\text{F} = 10^{-6}\,\text{F}$).

Figure 5.1.3(a) shows the symbol which is used to represent capacitors in circuits. For a polarized fixed capacitor which has a definite polarity, Figure 5.1.3(b) is sometimes used.

Figure 5.1.3 Capacitor symbols.

5.2 Calculation of Capacitance

Let's see how capacitance can be computed in systems with simple geometry.

Example 5.1: Parallel-Plate Capacitor

Consider two metallic plates of equal area *A* separated by a distance *d*, as shown in Figure 5.2.1 below. The top plate carries a charge $+Q$ while the bottom plate carries a charge $-Q$. The charging of the plates can be accomplished by means of a battery which produces a potential difference. Find the capacitance of the system.

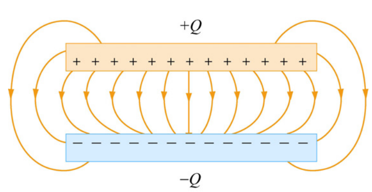


Figure 5.2.1 The electric field between the plates of a parallel-plate capacitor

Solution:

To find the capacitance C, we first need to know the electric field between the plates. A real capacitor is finite in size. Thus, the electric field lines at the edge of the plates are not straight lines, and the field is not contained entirely between the plates. This is known as *edge effects*, and the non-uniform fields near the edge are called the *fringing fields*. In Figure 5.2.1 the field lines are drawn by taking into consideration edge effects. However, in what follows, we shall ignore such effects and assume an idealized situation, where field lines between the plates are straight lines.

In the limit where the plates are infinitely large, the system has planar symmetry and we can calculate the electric field everywhere using Gauss's law given in Eq. (4.2.5):

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{q_{\text{enc}}}{\varepsilon_0}$$

By choosing a Gaussian "pillbox" with cap area A' to enclose the charge on the positive plate (see Figure 5.2.2), the electric field in the region between the plates is

$$EA' = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\sigma A'}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{\sigma}{\varepsilon_0} \tag{5.2.1}$$

The same result has also been obtained in Section 4.8.1 using superposition principle.

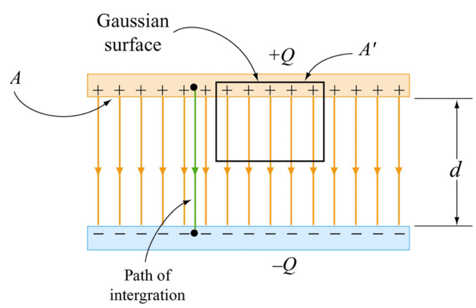


Figure 5.2.2 Gaussian surface for calculating the electric field between the plates.

The potential difference between the plates is

$$\Delta V = V_- - V_+ = -\int_+^- \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} = -Ed \tag{5.2.2}$$

where we have taken the path of integration to be a straight line from the positive plate to the negative plate following the field lines (Figure 5.2.2). Since the electric field lines are always directed from higher potential to lower potential, $V_- < V_+$. However, in computing the capacitance C, the relevant quantity is the magnitude of the potential difference:

$$|\Delta V| = Ed \tag{5.2.3}$$

and its sign is immaterial. From the definition of capacitance, we have

$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{\varepsilon_0 A}{d} \qquad \text{(parallel plate)}} \tag{5.2.4}$$

Note that C depends only on the geometric factors A and d. The capacitance C increases linearly with the area A since for a given potential difference ΔV, a bigger plate can hold more charge. On the other hand, C is inversely proportional to d, the distance of separation because the smaller the value of d, the smaller the potential difference $|\Delta V|$ for a fixed Q.

Interactive Simulation 5.1: **Parallel-Plate Capacitor**

This simulation shown in Figure 5.2.3 illustrates the interaction of charged particles inside the two plates of a capacitor.

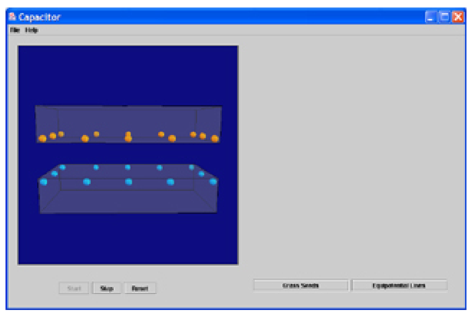


Figure 5.2.3 Charged particles interacting inside the two plates of a capacitor.

Each plate contains twelve charges interacting via Coulomb force, where one plate contains positive charges and the other contains negative charges. Because of their mutual repulsion, the particles in each plate are compelled to maximize the distance between one another, and thus spread themselves evenly around the outer edge of their enclosure. However, the particles in one plate are attracted to the particles in the other, so they attempt to minimize the distance between themselves and their oppositely charged correspondents. Thus, they distribute themselves along the surface of their bounding box closest to the other plate.

Example 5.2: Cylindrical Capacitor

Consider next a solid cylindrical conductor of radius a surrounded by a coaxial cylindrical shell of inner radius b, as shown in Figure 5.2.4. The length of both cylinders is L and we take this length to be much larger than $b-a$, the separation of the cylinders, so that edge effects can be neglected. The capacitor is charged so that the inner cylinder has charge $+Q$ while the outer shell has a charge $-Q$. What is the capacitance?

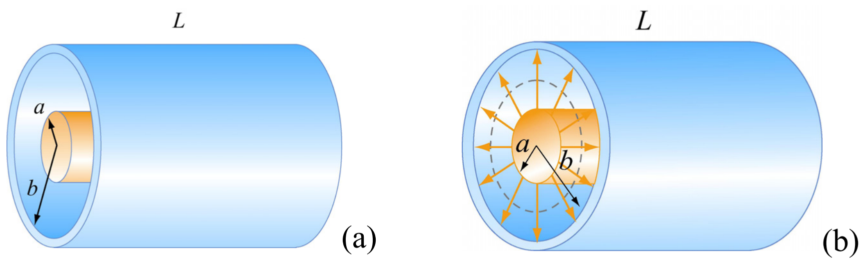


Figure 5.2.4 (a) A cylindrical capacitor. (b) End view of the capacitor. The electric field is non-vanishing only in the region $a < r < b$.

Solution:

To calculate the capacitance, we first compute the electric field everywhere. Due to the cylindrical symmetry of the system, we choose our Gaussian surface to be a coaxial cylinder with length $\ell < L$ and radius r where $a < r < b$. Using Gauss's law, we have

$$\oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = E\left(2\pi r\ell\right) = \frac{\lambda\ell}{\varepsilon_0} \quad \Rightarrow \quad E = \frac{\lambda}{2\pi\varepsilon_0 r} \tag{5.2.5}$$

where $\lambda = Q/L$ is the charge per unit length. Notice that the electric field is non-vanishing only in the region $a < r < b$. For $r < a$, the enclosed charge is $q_{\text{enc}} = 0$ since any net charge in a conductor must reside on its surface. Similarly, for $r > b$, the enclosed charge is $q_{\text{enc}} = \lambda\ell - \lambda\ell = 0$ since the Gaussian surface encloses equal but opposite charges from both conductors.

The potential difference is given by

$$\Delta V = V_b - V_a = -\int_a^b E_r\,dr = -\frac{\lambda}{2\pi\varepsilon_0}\int_a^b \frac{dr}{r} = -\frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{b}{a}\right) \tag{5.2.6}$$

where we have chosen the integration path to be along the direction of the electric field lines. As expected, the outer conductor with negative charge has a lower potential. This gives

$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{\lambda L}{\lambda \ln(b/a)/2\pi\varepsilon_0} = \frac{2\pi\varepsilon_0 L}{\ln(b/a)}} \tag{5.2.7}$$

Once again, we see that the capacitance C depends only on the geometrical factors, L, a and b.

Example 5.3: Spherical Capacitor

As a third example, let's consider a spherical capacitor which consists of two concentric spherical shells of radii a and b, as shown in Figure 5.2.5. The inner shell has a charge $+Q$ uniformly distributed over its surface, and the outer shell an equal but opposite charge $-Q$. What is the capacitance of this configuration?

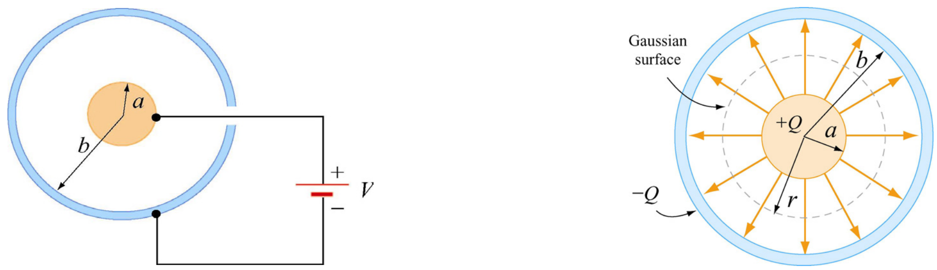


Figure 5.2.5 (a) spherical capacitor with two concentric spherical shells of radii a and b. (b) Gaussian surface for calculating the electric field.

Solution:

The electric field is non-vanishing only in the region $a < r < b$. Using Gauss's law, we obtain

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_r A = E_r \left(4\pi r^2\right) = \frac{Q}{\varepsilon_0} \tag{5.2.8}$$

or

$$E_r = \frac{1}{4\pi\varepsilon_o} \frac{Q}{r^2} \tag{5.2.9}$$

Therefore, the potential difference between the two conducting shells is:

$$\Delta V = V_b - V_a = -\int_a^b E_r dr = -\frac{Q}{4\pi\varepsilon_0}\int_a^b \frac{dr}{r^2} = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{a}-\frac{1}{b}\right) = -\frac{Q}{4\pi\varepsilon_0}\left(\frac{b-a}{ab}\right) \tag{5.2.10}$$

which yields

$$\boxed{C = \frac{Q}{|\Delta V|} = 4\pi\varepsilon_0\left(\frac{ab}{b-a}\right)} \tag{5.2.11}$$

Again, the capacitance C depends only on the physical dimensions, a and b.

An "isolated" conductor (with the second conductor placed at infinity) also has a capacitance. In the limit where $b \to \infty$, the above equation becomes

$$\lim_{b\to\infty} C = \lim_{b\to\infty} 4\pi\varepsilon_0 \left(\frac{ab}{b-a}\right) = \lim_{b\to\infty} 4\pi\varepsilon_0 \frac{a}{\left(1-\dfrac{a}{b}\right)} = 4\pi\varepsilon_0 a \tag{5.2.12}$$

Thus, for a single isolated spherical conductor of radius *R*, the capacitance is

$$C = 4\pi\varepsilon_0 R \tag{5.2.13}$$

The above expression can also be obtained by noting that a conducting sphere of radius *R* with a charge *Q* uniformly distributed over its surface has $V = Q/4\pi\varepsilon_0 R$, using infinity as the reference point having zero potential, $V(\infty) = 0$. This gives

$$C = \frac{Q}{|\Delta V|} = \frac{Q}{Q/4\pi\varepsilon_0 R} = 4\pi\varepsilon_0 R \tag{5.2.14}$$

As expected, the capacitance of an isolated charged sphere only depends on its geometry, namely, the radius *R*.

5.3 Capacitors in Electric Circuits

A capacitor can be charged by connecting the plates to the terminals of a battery, which are maintained at a potential difference ΔV called the *terminal voltage*.

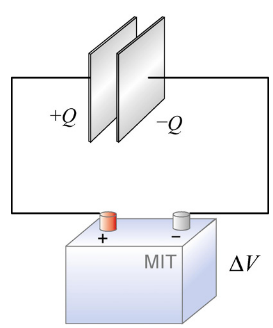


Figure 5.3.1 Charging a capacitor.

The connection results in sharing the charges between the terminals and the plates. For example, the plate that is connected to the (positive) negative terminal will acquire some (positive) negative charge. The sharing causes a momentary reduction of charges on the terminals, and a decrease in the terminal voltage. Chemical reactions are then triggered to transfer more charge from one terminal to the other to compensate for the loss of charge to the capacitor plates, and maintain the terminal voltage at its initial level. The battery could thus be thought of as a charge pump that brings a charge Q from one plate to the other.

5.3.1 Parallel Connection

Suppose we have two capacitors C_1 with charge Q_1 and C_2 with charge Q_2 that are connected in parallel, as shown in Figure 5.3.2.

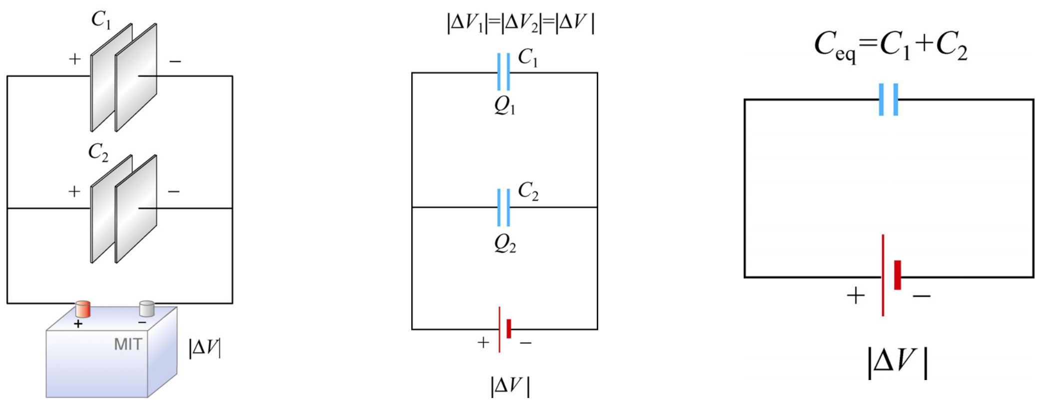


Figure 5.3.2 Capacitors in parallel and an equivalent capacitor.

The left plates of both capacitors C_1 and C_2 are connected to the positive terminal of the battery and have the same electric potential as the positive terminal. Similarly, both right plates are negatively charged and have the same potential as the negative terminal. Thus, the potential difference $|\Delta V|$ is the same across each capacitor. This gives

$$C_1 = \frac{Q_1}{|\Delta V|}, \qquad C_2 = \frac{Q_2}{|\Delta V|} \tag{5.3.1}$$

These two capacitors can be replaced by a single equivalent capacitor C_{eq} with a total charge Q supplied by the battery. However, since Q is shared by the two capacitors, we must have

$$Q = Q_1 + Q_2 = C_1|\Delta V| + C_2|\Delta V| = (C_1 + C_2)|\Delta V| \tag{5.3.2}$$

The equivalent capacitance is then seen to be given by

$$C_{eq} = \frac{Q}{|\Delta V|} = C_1 + C_2 \tag{5.3.3}$$

Thus, capacitors that are connected in parallel add. The generalization to any number of capacitors is

$$\boxed{C_{eq} = C_1 + C_2 + C_3 + \cdots + C_N = \sum_{i=1}^{N} C_i \qquad \text{(parallel)}} \tag{5.3.4}$$

5.3.2 Series Connection

Suppose two initially uncharged capacitors C_1 and C_2 are connected in series, as shown in Figure 5.3.3. A potential difference $|\Delta V|$ is then applied across both capacitors. The left plate of capacitor 1 is connected to the positive terminal of the battery and becomes positively charged with a charge $+Q$, while the right plate of capacitor 2 is connected to the negative terminal and becomes negatively charged with charge $-Q$ as electrons flow in. What about the inner plates? They were initially uncharged; now the outside plates each attract an equal and opposite charge. So the right plate of capacitor 1 will acquire a charge $-Q$ and the left plate of capacitor $+Q$.

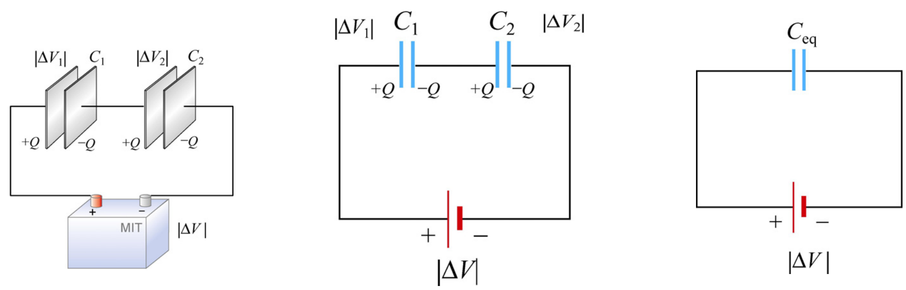


Figure 5.3.3 Capacitors in series and an equivalent capacitor

The potential differences across capacitors C_1 and C_2 are

$$|\Delta V_1| = \frac{Q}{C_1}, \quad |\Delta V_2| = \frac{Q}{C_2} \tag{5.3.5}$$

respectively. From Figure 5.3.3, we see that the total potential difference is simply the sum of the two individual potential differences:

$$|\Delta V| = |\Delta V_1| + |\Delta V_2| \tag{5.3.6}$$

In fact, the total potential difference across any number of capacitors in series connection is equal to the sum of potential differences across the individual capacitors. These two capacitors can be replaced by a single equivalent capacitor $C_{\text{eq}} = Q/|\Delta V|$. Using the fact that the potentials add in series,

$$\frac{Q}{C_{\text{eq}}} = \frac{Q}{C_1} + \frac{Q}{C_2}$$

and so the equivalent capacitance for two capacitors in series becomes

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} \tag{5.3.7}$$

The generalization to any number of capacitors connected in series is

$$\boxed{\frac{1}{C_{\text{eq}}}=\frac{1}{C_1}+\frac{1}{C_2}+\cdots+\frac{1}{C_N}=\sum_{i=1}^{N}\frac{1}{C_i} \qquad (\text{series})} \tag{5.3.8}$$

Example 5.4: Equivalent Capacitance

Find the equivalent capacitance for the combination of capacitors shown in Figure 5.3.4(a)

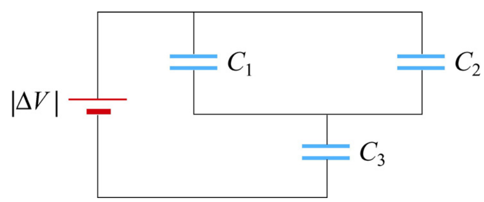


Figure 5.3.4 (a) Capacitors connected in series and in parallel

Solution:

Since C_1 and C_2 are connected in parallel, their equivalent capacitance C_{12} is given by

$$C_{12}=C_1+C_2$$

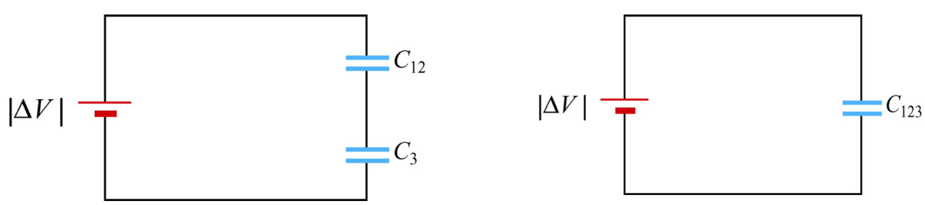


Figure 5.3.4 (b) and (c) Equivalent circuits.

Now capacitor C_{12} is in series with C_3, as seen from Figure 5.3.4(b). So, the equivalent capacitance C_{123} is given by

$$\frac{1}{C_{123}}=\frac{1}{C_{12}}+\frac{1}{C_3}$$

or

$$\boxed{C_{123}=\frac{C_{12}C_3}{C_{12}+C_3}=\frac{(C_1+C_2)C_3}{C_1+C_2+C_3}}$$

5.4 Storing Energy in a Capacitor

As discussed in the introduction, capacitors can be used to stored electrical energy. The amount of energy stored is equal to the work done to charge it. During the charging process, the battery does work to remove charges from one plate and deposit them onto the other.

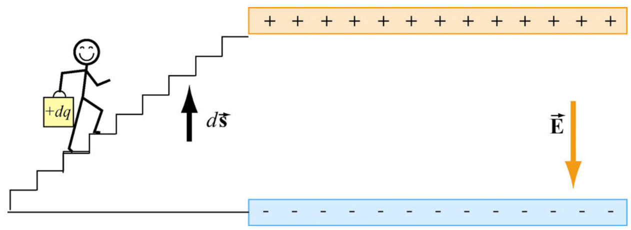


Figure 5.4.1 Work is done by an external agent in bringing $+dq$ from the negative plate and depositing the charge on the positive plate.

Let the capacitor be initially uncharged. In each plate of the capacitor, there are many negative and positive charges, but the number of negative charges balances the number of positive charges, so that there is no net charge, and therefore no electric field between the plates. We have a magic bucket and a set of stairs from the bottom plate to the top plate (Figure 5.4.1).

We start out at the bottom plate, fill our magic bucket with a charge $+dq$, carry the bucket up the stairs and dump the contents of the bucket on the top plate, charging it up positive to charge $+dq$. However, in doing so, the bottom plate is now charged to $-dq$. Having emptied the bucket of charge, we now descend the stairs, get another bucketful of charge $+dq$, go back up the stairs and dump that charge on the top plate. We then repeat this process over and over. In this way we build up charge on the capacitor, and create electric field where there was none initially.

Suppose the amount of charge on the top plate at some instant is $+q$, and the potential difference between the two plates is $|\Delta V| = q/C$. To dump another bucket of charge $+dq$ on the top plate, the amount of work done to overcome electrical repulsion is $dW = |\Delta V|\,dq$. If at the end of the charging process, the charge on the top plate is $+Q$, then the total amount of work done in this process is

$$W = \int_0^Q dq\,|\Delta V| = \int_0^Q dq\,\frac{q}{C} = \frac{1}{2}\frac{Q^2}{C} \tag{5.4.1}$$

This is equal to the electrical potential energy U_E of the system:

$$\boxed{U_E = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}Q\,|\Delta V| = \frac{1}{2}C\,|\Delta V|^2} \tag{5.4.2}$$

5.4.1 Energy Density of the Electric Field

One can think of the energy stored in the capacitor as being stored in the electric field itself. In the case of a parallel-plate capacitor, with $C = \varepsilon_0 A/d$ and $|\Delta V| = Ed$, we have

$$U_E = \frac{1}{2}C|\Delta V|^2 = \frac{1}{2}\frac{\varepsilon_0 A}{d}\left(Ed\right)^2 = \frac{1}{2}\varepsilon_0 E^2\left(Ad\right) \tag{5.4.3}$$

Since the quantity *Ad* represents the volume between the plates, we can define the electric energy density as

$$\boxed{u_E = \frac{U_E}{\text{Volume}} = \frac{1}{2}\varepsilon_0 E^2} \tag{5.4.4}$$

Note that u_E is proportional to the square of the electric field. Alternatively, one may obtain the energy stored in the capacitor from the point of view of external work. Since the plates are oppositely charged, force must be applied to maintain a constant separation between them. From Eq. (4.4.7), we see that a small patch of charge $\Delta q = \sigma(\Delta A)$ experiences an attractive force $\Delta F = \sigma^2(\Delta A)/2\varepsilon_0$. If the total area of the plate is *A*, then an external agent must exert a force $F_{\text{ext}} = \sigma^2 A/2\varepsilon_0$ to pull the two plates apart. Since the electric field strength in the region between the plates is given by $E = \sigma/\varepsilon_0$, the external force can be rewritten as

$$F_{\text{ext}} = \frac{\varepsilon_0}{2}E^2 A \tag{5.4.5}$$

Note that F_{ext} is independent of d. The total amount of work done externally to separate the plates by a distance *d* is then

$$W_{\text{ext}} = \int \vec{\mathbf{F}}_{\text{ext}} \cdot d\vec{\mathbf{s}} = F_{\text{ext}} d = \left(\frac{\varepsilon_0 E^2 A}{2}\right) d \tag{5.4.6}$$

consistent with Eq. (5.4.3). Since the potential energy of the system is equal to the work done by the external agent, we have $u_E = W_{\text{ext}}/Ad = \varepsilon_0 E^2/2$. In addition, we note that the expression for u_E is identical to Eq. (4.4.8) in Chapter 4. Therefore, the electric energy density u_E can also be interpreted as electrostatic pressure *P*.

Interactive Simulation 5.2: Charge Placed between Capacitor Plates

This applet shown in Figure 5.4.2 is a simulation of an experiment in which an aluminum sphere sitting on the bottom plate of a capacitor is lifted to the top plate by the electrostatic force generated as the capacitor is charged. We have placed a non-conducting barrier just below the upper plate to prevent the sphere from touching it and discharging.

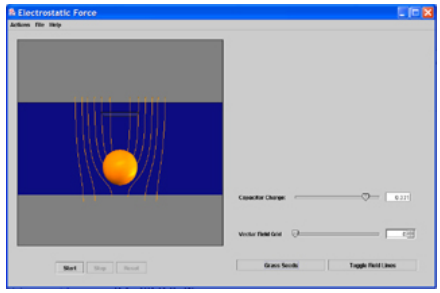


Figure 5.4.2 Electrostatic force experienced by an aluminum sphere placed between the plates of a parallel-plate capacitor.

While the sphere is in contact with the bottom plate, the charge density of the bottom of the sphere is the same as that of the lower plate. Thus, as the capacitor is charged, the charge density on the sphere increases proportional to the potential difference between the plates. In addition, energy flows in to the region between the plates as the electric field builds up. This can be seen in the motion of the electric field lines as they move from the edge to the center of the capacitor.

As the potential difference between the plates increases, the sphere feels an increasing attraction towards the top plate, indicated by the increasing tension in the field as more field lines "attach" to it. Eventually this tension is enough to overcome the downward force of gravity, and the sphere is lifted. Once separated from the lower plate, the sphere charge density no longer increases, and it feels both an attractive force towards the upper plate (whose charge is roughly opposite that of the sphere) and a repulsive force from the lower one (whose charge is roughly equal to that of the sphere). The result is a net force upwards.

Example 5.5: Electric Energy Density of Dry Air

The breakdown field strength at which dry air loses its insulating ability and allows a discharge to pass through is $E_b = 3\times 10^6\,\text{V/m}$. At this field strength, the electric energy density is:

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{1}{2}\left(8.85\times 10^{-12}\,\text{C}^2/\text{N}\cdot\text{m}^2\right)\left(3\times 10^6\,\text{V/m}\right)^2 = 40\ \text{J/m}^3 \qquad (5.4.7)$$

Example 5.6: Energy Stored in a Spherical Shell

Find the energy stored in a metallic spherical shell of radius *a* and charge *Q*.

Solution:

The electric field associated of a spherical shell of radius a is (Example 4.3)

$$\vec{\mathbf{E}} = \begin{cases} \dfrac{Q}{4\pi\varepsilon_0 r^2}\,\hat{\mathbf{r}}, & r > a \\ \vec{\mathbf{0}}, & r < a \end{cases} \tag{5.4.8}$$

The corresponding energy density is

$$u_E = \frac{1}{2}\varepsilon_0 E^2 = \frac{Q^2}{32\pi^2\varepsilon_0 r^4} \tag{5.4.9}$$

outside the sphere, and zero inside. Since the electric field is non-vanishing outside the spherical shell, we must integrate over the entire region of space from $r = a$ to $r = \infty$. In spherical coordinates, with $dV = 4\pi r^2 dr$, we have

$$U_E = \int_a^\infty \left(\frac{Q^2}{32\pi^2\varepsilon_0 r^4}\right) 4\pi r^2 dr = \frac{Q^2}{8\pi\varepsilon_0}\int_a^\infty \frac{dr}{r^2} = \frac{Q^2}{8\pi\varepsilon_0 a} = \frac{1}{2}QV \tag{5.4.10}$$

where $V = Q/4\pi\varepsilon_0 a$ is the electric potential on the surface of the shell, with $V(\infty) = 0$. We can readily verify that the energy of the system is equal to the work done in charging the sphere. To show this, suppose at some instant the sphere has charge q and is at a potential $V = q/4\pi\varepsilon_0 a$. The work required to add an additional charge dq to the system is $dW = Vdq$. Thus, the total work is

$$W = \int dW = \int V dq = \int_0^Q dq\left(\frac{q}{4\pi\varepsilon_0 a}\right) = \frac{Q^2}{8\pi\varepsilon_0 a} \tag{5.4.11}$$

5.5 Dielectrics

In many capacitors there is an insulating material such as paper or plastic between the plates. Such material, called a dielectric, can be used to maintain a physical separation of the plates. Since dielectrics break down less readily than air, charge leakage can be minimized, especially when high voltage is applied.

Experimentally it was found that capacitance C increases when the space between the conductors is filled with dielectrics. To see how this happens, suppose a capacitor has a capacitance C_0 when there is no material between the plates. When a dielectric material is inserted to completely fill the space between the plates, the capacitance increases to

$$C = \kappa_e C_0 \tag{5.5.1}$$

where κ_e is called the dielectric constant. In the Table below, we show some dielectric materials with their dielectric constant. Experiments indicate that all dielectric materials have $\kappa_e > 1$. Note that every dielectric material has a characteristic dielectric strength which is the maximum value of electric field before breakdown occurs and charges begin to flow.

Material	κ_e	Dielectric strength $(10^6\,\mathrm{V/m})$
Air	1.00059	3
Paper	3.7	16
Glass	4–6	9
Water	80	–

The fact that capacitance increases in the presence of a dielectric can be explained from a molecular point of view. We shall show that κ_e is a measure of the dielectric response to an external electric field. There are two types of dielectrics. The first type is polar dielectrics, which are dielectrics that have permanent electric dipole moments. An example of this type of dielectric is water.

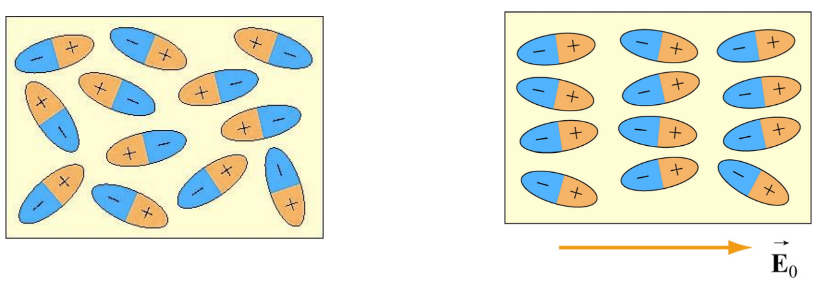


Figure 5.5.1 Orientations of polar molecules when (a) $\vec{\mathbf{E}}_0 = \vec{\mathbf{0}}$ and (b) $\vec{\mathbf{E}}_0 \neq 0$.

As depicted in Figure 5.5.1, the orientation of polar molecules is random in the absence of an external field. When an external electric field $\vec{\mathbf{E}}_0$ is present, a torque is set up and causes the molecules to align with $\vec{\mathbf{E}}_0$. However, the alignment is not complete due to random thermal motion. The aligned molecules then generate an electric field that is opposite to the applied field but smaller in magnitude.

The second type of dielectrics is the non-polar dielectrics, which are dielectrics that do not possess permanent electric dipole moment. Electric dipole moments can be induced by placing the materials in an externally applied electric field.

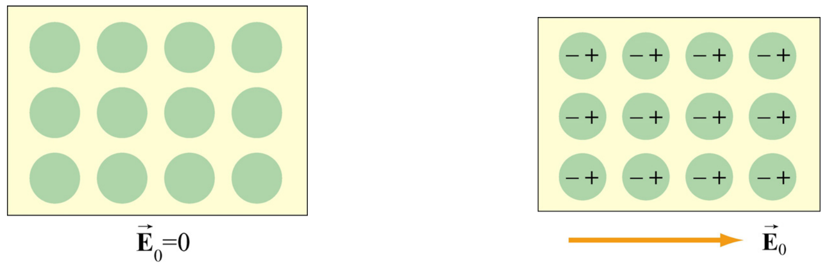


Figure 5.5.2 Orientations of non-polar molecules when (a) $\vec{\mathbf{E}}_0 = \vec{\mathbf{0}}$ and (b) $\vec{\mathbf{E}}_0 \neq \vec{\mathbf{0}}$.

Figure 5.5.2 illustrates the orientation of non-polar molecules with and without an external field $\vec{\mathbf{E}}_0$. The induced surface charges on the faces produces an electric field $\vec{\mathbf{E}}_P$ in the direction opposite to $\vec{\mathbf{E}}_0$, leading to $\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_P$, with $|\vec{\mathbf{E}}| < |\vec{\mathbf{E}}_0|$. Below we show how the induced electric field $\vec{\mathbf{E}}_P$ is calculated.

5.5.1 Polarization

We have shown that dielectric materials consist of many permanent or induced electric dipoles. One of the concepts crucial to the understanding of dielectric materials is the average electric field produced by many little electric dipoles which are all aligned. Suppose we have a piece of material in the form of a cylinder with area A and height h, as shown in Figure 5.5.3, and that it consists of N electric dipoles, each with electric dipole moment $\vec{\mathbf{p}}$ spread uniformly throughout the volume of the cylinder.

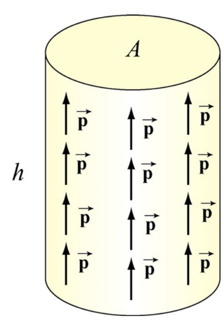


Figure 5.5.3 A cylinder with uniform dipole distribution.

We furthermore assume for the moment that all of the electric dipole moments $\vec{\mathbf{p}}$ are aligned with the axis of the cylinder. Since each electric dipole has its own electric field associated with it, in the absence of any external electric field, if we average over all the individual fields produced by the dipole, what is the average electric field just due to the presence of the aligned dipoles?

To answer this question, let us define the polarization vector $\vec{\mathbf{P}}$ to be the net electric dipole moment vector per unit volume:

$$\vec{\mathbf{P}} = \frac{1}{\text{Volume}} \sum_{i=1}^{N} \vec{\mathbf{p}}_i \tag{5.5.2}$$

In the case of our cylinder, where all the dipoles are perfectly aligned, the magnitude of $\vec{\mathbf{P}}$ is equal to

$$P = \frac{Np}{Ah} \tag{5.5.3}$$

and the direction of $\vec{\mathbf{P}}$ is parallel to the aligned dipoles.

Now, what is the average electric field these dipoles produce? The key to figuring this out is realizing that the situation shown in Figure 5.5.4(a) is equivalent that shown in Figure 5.5.4(b), where all the little ± charges associated with the electric dipoles in the interior of the cylinder are replaced with two equivalent charges, $\pm Q_P$, on the top and bottom of the cylinder, respectively.

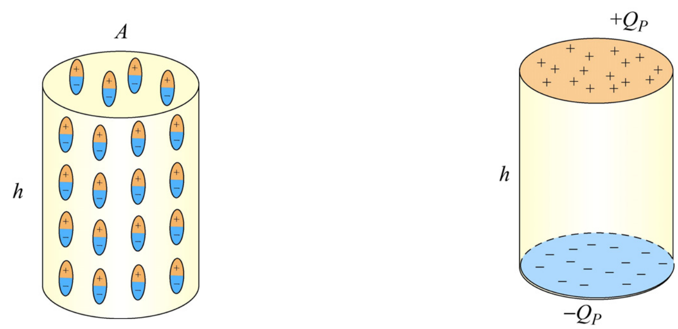


Figure 5.5.4 (a) A cylinder with uniform dipole distribution. (b) Equivalent charge distribution.

The equivalence can be seen by noting that in the interior of the cylinder, positive charge at the top of any one of the electric dipoles is *canceled* on average by the negative charge of the dipole just above it. The only place where cancellation does not take place is for electric dipoles at the top of the cylinder, since there are no adjacent dipoles further up. Thus the interior of the cylinder appears uncharged in an average sense (averaging over many dipoles), whereas the top surface of the cylinder appears to carry a net positive charge. Similarly, the bottom surface of the cylinder will appear to carry a net negative charge.

How do we find an expression for the equivalent charge Q_P in terms of quantities we know? The simplest way is to require that the electric dipole moment Q_P produces, $Q_P h$, is equal to the total electric dipole moment of all the little electric dipoles. This gives $Q_P h = Np$, or

$$Q_P = \frac{Np}{h} \tag{5.5.4}$$

To compute the electric field produced by Q_P, we note that the equivalent charge distribution resembles that of a parallel-plate capacitor, with an equivalent surface charge density σ_P that is equal to the magnitude of the polarization:

$$\sigma_P = \frac{Q_P}{A} = \frac{Np}{Ah} = P \tag{5.5.5}$$

Note that the SI units of P are $(\mathrm{C\cdot m})/\mathrm{m}^3$, or $\mathrm{C/m}^2$, which is the same as the surface charge density. In general if the polarization vector makes an angle θ with $\hat{\mathbf{n}}$, the outward normal vector of the surface, the surface charge density would be

$$\sigma_P = \vec{\mathbf{P}}\cdot\hat{\mathbf{n}} = P\cos\theta \tag{5.5.6}$$

Thus, our equivalent charge system will produce an average electric field of magnitude $E_P = P/\varepsilon_0$. Since the direction of this electric field is *opposite* to the direction of $\vec{\mathbf{P}}$, in vector notation, we have

$$\vec{\mathbf{E}}_P = -\vec{\mathbf{P}}/\varepsilon_0 \tag{5.5.7}$$

Thus, the average electric field of all these dipoles is opposite to the direction of the dipoles themselves. It is important to realize that this is just the *average* field due to all the dipoles. If we go close to any individual dipole, we will see a very different field.

We have assumed here that all our electric dipoles are aligned. In general, if these dipoles are randomly oriented, then the polarization $\vec{\mathbf{P}}$ given in Eq. (5.5.2) will be zero, and there will be no average field due to their presence. If the dipoles have some tendency toward a preferred orientation, then $\vec{\mathbf{P}} \neq \vec{\mathbf{0}}$, leading to a non-vanishing average field $\vec{\mathbf{E}}_P$.

Let us now examine the effects of introducing dielectric material into a system. We shall first assume that the atoms or molecules comprising the dielectric material have a *permanent* electric dipole moment. If left to themselves, these permanent electric dipoles in a dielectric material never line up spontaneously, so that in the absence of any applied external electric field, $\vec{\mathbf{P}} = \vec{\mathbf{0}}$ due to the random alignment of dipoles, and the average electric field $\vec{\mathbf{E}}_P$ is zero as well. However, when we place the dielectric material in an external field $\vec{\mathbf{E}}_0$, the dipoles will experience a torque $\vec{\boldsymbol{\tau}} = \vec{\mathbf{p}}\times\vec{\mathbf{E}}_0$ that tends to align the dipole vectors $\vec{\mathbf{p}}$ with $\vec{\mathbf{E}}_0$. The effect is a net polarization $\vec{\mathbf{P}}$ parallel to $\vec{\mathbf{E}}_0$, and therefore an average electric field of the dipoles $\vec{\mathbf{E}}_P$ *anti-parallel* to $\vec{\mathbf{E}}_0$, i.e., that will tend to *reduce* the total electric field strength below $\vec{\mathbf{E}}_0$. The total electric field $\vec{\mathbf{E}}$ is the sum of these two fields:

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_P = \vec{\mathbf{E}}_0 - \vec{\mathbf{P}} / \varepsilon_0 \quad (5.5.8)$$

In most cases, the polarization $\vec{\mathbf{P}}$ is not only in the same direction as $\vec{\mathbf{E}}_0$, but also linearly proportional to $\vec{\mathbf{E}}_0$ (and hence $\vec{\mathbf{E}}$.) This is reasonable because without the external field $\vec{\mathbf{E}}_0$ there would be no alignment of dipoles and no polarization $\vec{\mathbf{P}}$. We write the linear relation between $\vec{\mathbf{P}}$ and $\vec{\mathbf{E}}$ as

$$\vec{\mathbf{P}} = \varepsilon_0 \chi_e \vec{\mathbf{E}} \quad (5.5.9)$$

where χ_e is called the *electric susceptibility*. Materials they obey this relation are *linear dielectrics*. Combing Eqs. (5.5.8) and (5.5.7) gives

$$\vec{\mathbf{E}}_0 = (1 + \chi_e)\vec{\mathbf{E}} = \kappa_e \vec{\mathbf{E}} \quad (5.5.10)$$

where

$$\kappa_e = (1 + \chi_e) \quad (5.5.11)$$

is the dielectric constant. The dielectric constant κ_e is always greater than one since $\chi_e > 0$. This implies

$$E = \frac{E_0}{\kappa_e} < E_0 \quad (5.5.12)$$

Thus, we see that the effect of dielectric materials is always to decrease the electric field below what it would otherwise be.

In the case of dielectric material where there are no permanent electric dipoles, a similar effect is observed because the presence of an external field $\vec{\mathbf{E}}_0$ induces electric dipole moments in the atoms or molecules. These induced electric dipoles are parallel to $\vec{\mathbf{E}}_0$, again leading to a polarization $\vec{\mathbf{P}}$ parallel to $\vec{\mathbf{E}}_0$, and a reduction of the total electric field strength.

5.5.2 Dielectrics without Battery

As shown in Figure 5.5.5, a battery with a potential difference $|\Delta V_0|$ across its terminals is first connected to a capacitor C_0, which holds a charge $Q_0 = C_0 |\Delta V_0|$. We then disconnect the battery, leaving $Q_0 = \text{const}$.

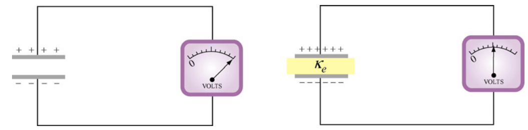


Figure 5.5.5 Inserting a dielectric material between the capacitor plates while keeping the charge Q_0 constant

If we then insert a dielectric between the plates, while keeping the charge constant, experimentally it is found that the potential difference decreases by a factor of κ_e:

$$|\Delta V| = \frac{|\Delta V_0|}{\kappa_e} \tag{5.5.13}$$

This implies that the capacitance is changed to

$$\boxed{C = \frac{Q}{|\Delta V|} = \frac{Q_0}{|\Delta V_0|/\kappa_e} = \kappa_e \frac{Q_0}{|\Delta V_0|} = \kappa_e C_0} \tag{5.5.14}$$

Thus, we see that the capacitance has increased by a factor of κ_e. The electric field within the dielectric is now

$$E = \frac{|\Delta V|}{d} = \frac{|\Delta V_0|/\kappa_e}{d} = \frac{1}{\kappa_e}\left(\frac{|\Delta V_0|}{d}\right) = \frac{E_0}{\kappa_e} \tag{5.5.15}$$

We see that in the presence of a dielectric, the electric field decreases by a factor of κ_e.

5.5.3 Dielectrics with Battery

Consider a second case where a battery supplying a potential difference $|\Delta V_0|$ remains connected as the dielectric is inserted. Experimentally, it is found (first by Faraday) that the charge on the plates is increased by a factor κ_e:

$$Q = \kappa_e Q_0 \tag{5.5.16}$$

where Q_0 is the charge on the plates in the absence of any dielectric.

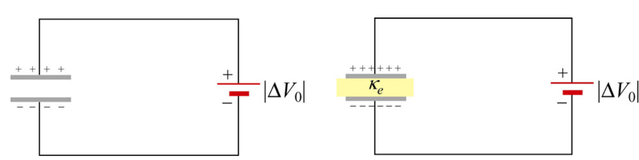


Figure 5.5.6 Inserting a dielectric material between the capacitor plates while maintaining a constant potential difference $|\Delta V_0|$.

The capacitance becomes

$$\boxed{C = \frac{Q}{|\Delta V_0|} = \frac{\kappa_e Q_0}{|\Delta V_0|} = \kappa_e C_0} \tag{5.5.17}$$

which is the same as the first case where the charge Q_0 is kept constant, but now the charge has increased.

5.5.4 Gauss's Law for Dielectrics

Consider again a parallel-plate capacitor shown in Figure 5.5.7:

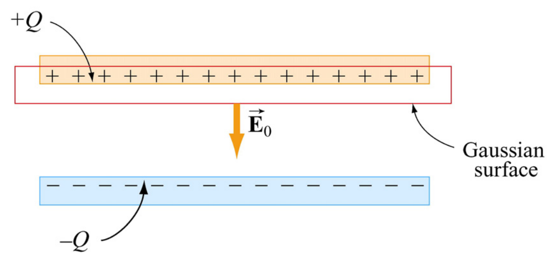


Figure 5.5.7 Gaussian surface in the absence of a dielectric.

When no dielectric is present, the electric field $\vec{\mathbf{E}}_0$ in the region between the plates can be found by using Gauss's law:

$$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_0 A = \frac{Q}{\varepsilon_0}, \quad \Rightarrow \quad E_0 = \frac{\sigma}{\varepsilon_0}$$

We have see that when a dielectric is inserted (Figure 5.5.8), there is an induced charge Q_P of opposite sign on the surface, and the net charge enclosed by the Gaussian surface is $Q - Q_P$.

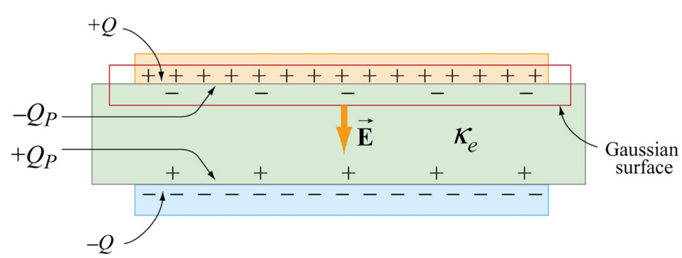


Figure 5.5.8 Gaussian surface in the presence of a dielectric.

Gauss's law becomes

$$\oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = \frac{Q - Q_P}{\varepsilon_0} \tag{5.5.18}$$

or

$$\boxed{E = \frac{Q - Q_P}{\varepsilon_0 A}} \tag{5.5.19}$$

However, we have just seen that the effect of the dielectric is to weaken the original field E_0 by a factor κ_e. Therefore,

$$E = \frac{E_0}{\kappa_e} = \frac{Q}{\kappa_e \varepsilon_0 A} = \frac{Q - Q_P}{\varepsilon_0 A} \tag{5.5.20}$$

from which the induced charge Q_P can be obtained as

$$Q_P = Q\left(1 - \frac{1}{\kappa_e}\right) \tag{5.5.21}$$

In terms of the surface charge density, we have

$$\sigma_P = \sigma\left(1 - \frac{1}{\kappa_e}\right) \tag{5.5.22}$$

Note that in the limit $\kappa_e = 1$, $Q_P = 0$ which corresponds to the case of no dielectric material.

Substituting Eq. (5.5.21) into Eq. (5.5.18), we see that Gauss's law with dielectric can be rewritten as

$$\oint\!\!\oint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = \frac{Q}{\kappa_e \varepsilon_0} = \frac{Q}{\varepsilon} \tag{5.5.23}$$

where $\varepsilon = \kappa_e \varepsilon_0$ is called the *dielectric permittivity*. Alternatively, we may also write

$$\oint\!\!\oint_S \vec{\mathbf{D}} \cdot d\vec{\mathbf{A}} = Q \tag{5.5.24}$$

where $\vec{\mathbf{D}} = \varepsilon_0 \kappa \vec{\mathbf{E}}$ is called the *electric displacement vector*.

Example 5.7: Capacitance with Dielectrics

A non-conducting slab of thickness t, area A and dielectric constant κ_e is inserted into the space between the plates of a parallel-plate capacitor with spacing d, charge Q and area A, as shown in Figure 5.5.9(a). The slab is not necessarily halfway between the capacitor plates. What is the capacitance of the system?

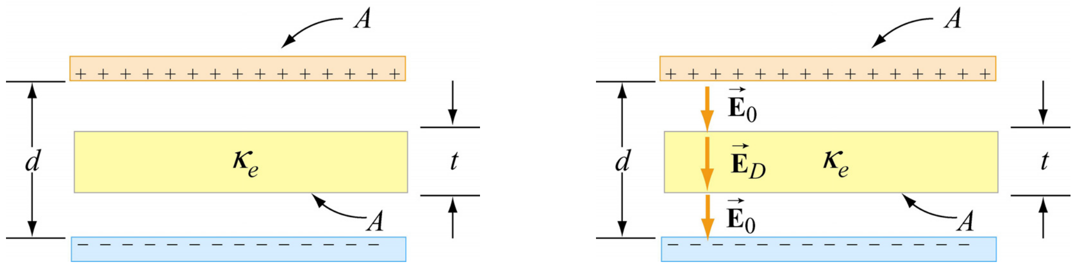


Figure 5.5.9 (a) Capacitor with a dielectric. (b) Electric field between the plates.

Solution:

To find the capacitance C, we first calculate the potential difference ΔV. We have already seen that in the absence of a dielectric, the electric field between the plates is given by $E_0 = Q/\varepsilon_0 A$, and $E_D = E_0/\kappa_e$ when a dielectric of dielectric constant κ_e is present, as shown in Figure 5.5.9(b). The potential can be found by integrating the electric field along a straight line from the top to the bottom plates:

$$\begin{aligned}\Delta V &= -\int_+^- E\,dl = -\Delta V_0 - \Delta V_D = -E_0\left(d-t\right) - E_D t = -\frac{Q}{A\varepsilon_0}\left(d-t\right) - \frac{Q}{A\varepsilon_0\kappa_e}t \\ &= -\frac{Q}{A\varepsilon_0}\left[d - t\left(1-\frac{1}{\kappa_e}\right)\right]\end{aligned} \tag{5.5.25}$$

where $\Delta V_D = E_D t$ is the potential difference between the two faces of the dielectric. This gives

$$C = \frac{Q}{|\Delta V|} = \frac{\varepsilon_0 A}{d - t\left(1-\dfrac{1}{\kappa_e}\right)} \tag{5.5.26}$$

It is useful to check the following limits:

(i) As $t \to 0$, *i.e.*, the thickness of the dielectric approaches zero, we have $C = \varepsilon_0 A/d = C_0$, which is the expected result for no dielectric.

(ii) As $\kappa_e \to 1$, we again have $C \to \varepsilon_0 A/d = C_0$, and the situation also correspond to the case where the dielectric is absent.

(iii) In the limit where $t \to d$, the space is filled with dielectric, we have $C \to \kappa_e \varepsilon_0 A / d = \kappa_e C_0$.

We also comment that the configuration is equivalent to two capacitors connected in series, as shown in Figure 5.5.10.

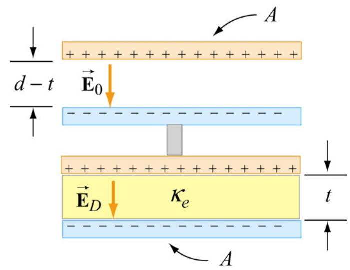


Figure 5.5.10 Equivalent configuration.

Using Eq. (5.3.8) for capacitors connected in series, the equivalent capacitance is

$$\frac{1}{C} = \frac{d-t}{\varepsilon_0 A} + \frac{t}{\kappa_e \varepsilon_0 A} \tag{5.5.28}$$

5.6 Creating Electric Fields

Animation 5.1: Creating an Electric Dipole

Electric fields are created by electric charge. If there is no electric charge present, and there never has been any electric charge present in the past, then there would be no electric field anywhere is space. How is electric field created and how does it come to fill up space? To answer this, consider the following scenario in which we go from the electric field being zero everywhere in space to an electric field existing everywhere in space.

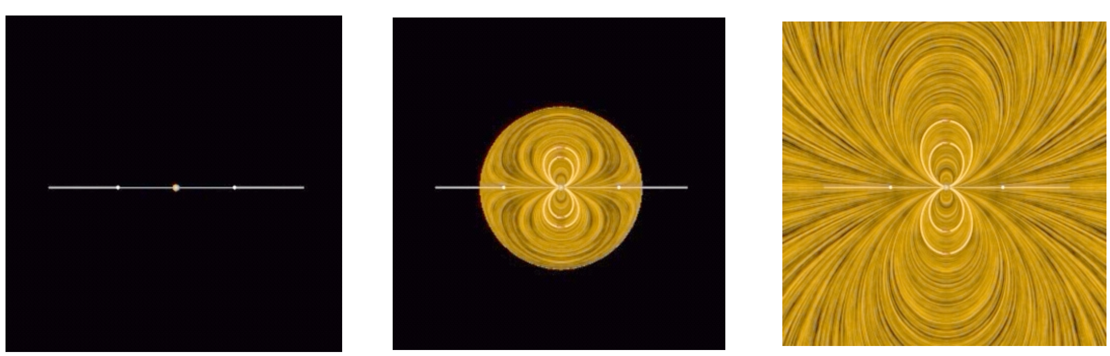


Figure 5.6.1 Creating an electric dipole. (a) Before any charge separation. (b) Just after the charges are separated. (c) A long time after the charges are separated.

Suppose we have a positive point charge sitting right on top of a negative electric charge, so that the total charge exactly cancels, and there is no electric field anywhere in space. Now let us pull these two charges apart slightly, so that they are separated by a small distance. If we allow them to sit at that distance for a long time, there will now be a charge imbalance – an electric dipole. The dipole will create an electric field.

Let us see how this creation of electric field takes place in detail. Figure 5.6.1 shows three frames of an animation of the process of separating the charges. In Figure 5.6.1(a), there is no charge separation, and the electric field is zero everywhere in space. Figure 5.6.1(b) shows what happens just after the charges are first separated. An expanding sphere of electric fields is observed. Figure 5.6.1(c) is a long time after the charges are separated (that is, they have been at a constant distance from another for a long time). An electric dipole has been created.

What does this sequence tell us? The following conclusions can be drawn:

(1) It is electric charge that generates electric field — no charge, no field.

(2) The electric field does not appear instantaneously in space everywhere as soon as there is unbalanced charge — the electric field propagates outward from its source at some finite speed. This speed will turn out to be the speed of light, as we shall see later.

(3) After the charge distribution settles down and becomes stationary, so does the field configuration. The initial field pattern associated with the time dependent separation of the charge is actually a burst of “electric dipole radiation.” We return to the subject of radiation at the end of this course. Until then, we will neglect radiation fields. The field configuration left behind after a long time is just the electric dipole pattern discussed above.

We note that the external agent who pulls the charges apart has to do work to keep them separate, since they attract each other as soon as they start to separate. Therefore, the external work done is to overcome the electrostatic attraction. In addition, the work also goes into providing the energy carried off by radiation, as well as the energy needed to set up the final stationary electric field that we see in Figure 5.6.1(c).

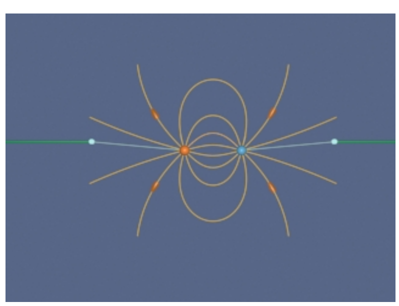


Figure 5.6.2 Creating the electric fields of two point charges by pulling apart two opposite charges initially on top of one another. We artificially terminate the field lines at a fixed distance from the charges to avoid visual confusion.

Finally, we ignore radiation and complete the process of separating our opposite point charges that we began in Figure 5.6.1. Figure 5.6.2 shows the complete sequence. When we finish and have moved the charges far apart, we see the characteristic radial field in the vicinity of a point charge.

Animation 5.2: Creating and Destroying Electric Energy

Let us look at the process of creating electric energy in a different context. We ignore energy losses due to radiation in this discussion. Figure 5.6.3 shows one frame of an animation that illustrates the following process.

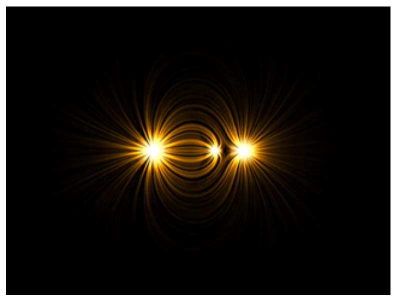


Figure 5.6.3 Creating and destroying electric energy.

We start out with five negative electric charges and five positive charges, all at the same point in space. Sine there is no net charge, there is no electric field. Now we move one of the positive charges at constant velocity from its initial position to a distance L away along the horizontal axis. After doing that, we move the second positive charge in the same manner to the position where the first positive charge sits. After doing that, we continue on with the rest of the positive charges in the same manner, until all the positive charges are sitting a distance L from their initial position along the horizontal axis. Figure 5.6.3 shows the field configuration during this process. We have color coded the "grass seeds" representation to represent the strength of the electric field. Very strong fields are white, very weak fields are black, and fields of intermediate strength are yellow.

Over the course of the "create" animation associated with Figure 5.6.3, the strength of the electric field grows as each positive charge is moved into place. The electric energy flows out from the path along which the charges move, and is being provided by the agent moving the charge against the electric field of the other charges. The work that this agent does to separate the charges against their electric attraction appears as energy in the electric field. We also have an animation of the opposite process linked to Figure 5.6.3. That is, we return in sequence each of the five positive charges to their original positions. At the end of this process we no longer have an electric field, because we no longer have an unbalanced electric charge.

On the other hand, over the course of the "destroy" animation associated with Figure 5.6.3, the strength of the electric field decreases as each positive charge is returned to its original position. The energy flows from the field back to the path along which the charges move, and is now being provided *to* the agent moving the charge at constant speed along the electric field of the other charges. The energy provided to that agent as we destroy the electric field is exactly the amount of energy that the agent put into creating the electric field in the first place, neglecting radiative losses (such losses are small if we move the charges at speeds small compared to the speed of light). This is a totally reversible process if we neglect such losses. That is, the amount of energy the agent puts into creating the electric field is exactly returned to that agent as the field is destroyed.

There is one final point to be made. Whenever electromagnetic energy is being created, an electric charge is moving (or being moved) against an electric field ($q\,\vec{\mathbf{v}}\cdot\vec{\mathbf{E}}<0$). Whenever electromagnetic energy is being destroyed, an electric charge is moving (or being moved) along an electric field ($q\,\vec{\mathbf{v}}\cdot\vec{\mathbf{E}}>0$). When we return to the creation and destruction of magnetic energy, we will find this rule holds there as well.

5.7 Summary

- A **capacitor** is a device that stores electric charge and potential energy. The **capacitance** C of a capacitor is the ratio of the charge stored on the capacitor plates to the the potential difference between them:

$$C=\frac{Q}{|\Delta V|}$$

System	Capacitance
Isolated charged sphere of radius R	$C=4\pi\varepsilon_0 R$
Parallel-plate capacitor of plate area A and plate separation d	$C=\varepsilon_0\dfrac{A}{d}$
Cylindrical capacitor of length L, inner radius a and outer radius b	$C=\dfrac{2\pi\varepsilon_0 L}{\ln(b/a)}$
Spherical capacitor with inner radius a and outer radius b	$C=4\pi\varepsilon_0\dfrac{ab}{(b-a)}$

- The equivalent capacitance of capacitors connected in parallel and in series are

$$C_{\mathrm{eq}}=C_1+C_2+C_3+\cdots \quad \text{(parallel)}$$

$$\frac{1}{C_{\mathrm{eq}}}=\frac{1}{C_1}+\frac{1}{C_2}+\frac{1}{C_3}+\cdots \quad \text{(series)}$$

- The work done in charging a capacitor to a charge Q is

$$U = \frac{Q^2}{2C} = \frac{1}{2}Q\,|\,\Delta V\,| = \frac{1}{2}C\,|\,\Delta V\,|^2$$

  This is equal to the amount of energy stored in the capacitor.

- The electric energy can also be thought of as stored in the electric field $\vec{\mathbf{E}}$. The **energy density** (energy per unit volume) is

$$u_E = \frac{1}{2}\varepsilon_0 E^2$$

  The energy density u_E is equal to the **electrostatic pressure** on a surface.

- When a dielectric material with **dielectric constant** κ_e is inserted into a capacitor, the capacitance increases by a factor κ_e:

$$C = \kappa_e C_0$$

- The **polarization** vector $\vec{\mathbf{P}}$ is the magnetic dipole moment per unit volume:

$$\vec{\mathbf{P}} = \frac{1}{V}\sum_{i=1}^{N}\vec{\mathbf{p}}_i$$

  The induced electric field due to polarization is

$$\vec{\mathbf{E}}_P = -\vec{\mathbf{P}}/\varepsilon_0$$

- In the presence of a dielectric with dielectric constant κ_e, the electric field becomes

$$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 + \vec{\mathbf{E}}_P = \vec{\mathbf{E}}_0/\kappa_e$$

  where $\vec{\mathbf{E}}_0$ is the electric field without dielectric.

5.8 Appendix: Electric Fields Hold Atoms Together

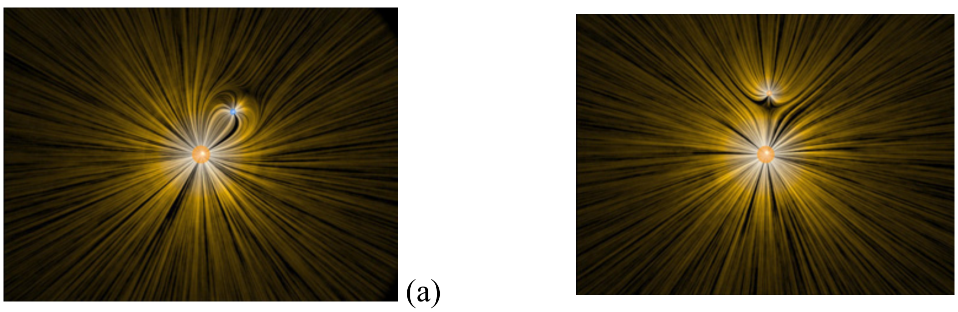
In this Appendix, we illustrate how electric fields are responsible for holding atoms together.

"…As our mental eye penetrates into smaller and smaller distances and shorter and shorter times, we find nature behaving so entirely differently from what we observe in visible and palpable bodies of our surroundings that no model shaped after our large-scale experiences can ever be "true". A completely satisfactory model of this type is not only practically inaccessible, but not even thinkable. Or, to be precise, we can, of course, think of it, but however we think it, it is wrong."

Erwin Schroedinger

5.8.1 Ionic and van der Waals Forces

Electromagnetic forces provide the "glue" that holds atoms together—that is, that keep electrons near protons and bind atoms together in solids. We present here a brief and very idealized model of how that happens from a semi-classical point of view.



(a) (b)

Figure 5.8.1 (a) A negative charge and (b) a positive charge moves past a massive positive particle at the origin and is deflected from its path by the stresses transmitted by the electric fields surrounding the charges.

Figure 5.8.1(a) illustrates the examples of the stresses transmitted by fields, as we have seen before. In Figure 5.8.1(a) we have a negative charge moving past a massive positive charge and being deflected toward that charge due to the attraction that the two charges feel. This attraction is mediated by the stresses transmitted by the electromagnetic field, and the simple interpretation of the interaction shown in Figure 5.8.1(b) is that the attraction is primarily due to a tension transmitted by the electric fields surrounding the charges.

In Figure 5.8.1(b) we have a positive charge moving past a massive positive charge and being deflected away from that charge due to the repulsion that the two charges feel. This repulsion is mediated by the stresses transmitted by the electromagnetic field, as we have discussed above, and the simple interpretation of the interaction shown in Figure 5.8.1(b) is that the repulsion is primarily due to a pressure transmitted by the electric fields surrounding the charges.

Consider the interaction of four charges of equal mass shown in Figure 5.8.2. Two of the charges are positively charged and two of the charges are negatively charged, and all have the same magnitude of charge. The particles interact via the Coulomb force.

We also introduce a quantum-mechanical “Pauli” force, which is always repulsive and becomes very important at small distances, but is negligible at large distances. The critical distance at which this repulsive force begins to dominate is about the radius of the spheres shown in Figure 5.8.2. This Pauli force is quantum mechanical in origin, and keeps the charges from collapsing into a point (i.e., it keeps a negative particle and a positive particle from sitting exactly on top of one another).

Additionally, the motion of the particles is damped by a term proportional to their velocity, allowing them to "settle down" into stable (or meta-stable) states.

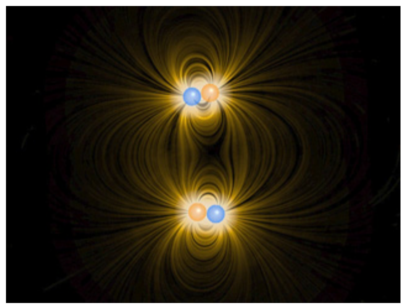


Figure 5.8.2 Four charges interacting via the Coulomb force, a repulsive Pauli force at close distances, with dynamic damping.

When these charges are allowed to evolve from the initial state, the first thing that happens (very quickly) is that the charges pair off into dipoles. This is a rapid process because the Coulomb attraction between unbalanced charges is very large. This process is called "ionic binding", and is responsible for the inter-atomic forces in ordinary table salt, NaCl. After the dipoles form, there is still an interaction between neighboring dipoles, but this is a much weaker interaction because the electric field of the dipoles falls off much faster than that of a single charge. This is because the net charge of the dipole is zero. When two opposite charges are close to one another, their electric fields “almost” cancel each other out.

Although in principle the dipole-dipole interaction can be either repulsive or attractive, in practice there is a torque that rotates the dipoles so that the dipole-dipole force is attractive. After a long time, this dipole-dipole attraction brings the two dipoles together in a bound state. The force of attraction between two dipoles is termed a “van der Waals” force, and it is responsible for intermolecular forces that bind some substances together into a solid.

Interactive Simulation 5.3: Collection of Charges in Two Dimensions

Figure 5.8.3 is an interactive two-dimensional ShockWave display that shows the same dynamical situation as in Figure 5.8.2 except that we have included a number of positive and negative charges, and we have eliminated the representation of the field so that we can interact with this simulation in real time. We start the charges at rest in random positions in space, and then let them evolve according to the forces that act on them (electrostatic attraction/repulsion, Pauli repulsion at very short distances, and a dynamic drag term proportional to velocity). The particles will eventually end up in a configuration in which the net force on any given particle is essentially zero. As we saw in the animation in Figure 5.8.3, generally the individual particles first pair off into dipoles and then slowly combine into larger structures. Rings and straight lines are the most common configurations, but by clicking and dragging particles around, the user can coax them into more complex meta-stable formations.

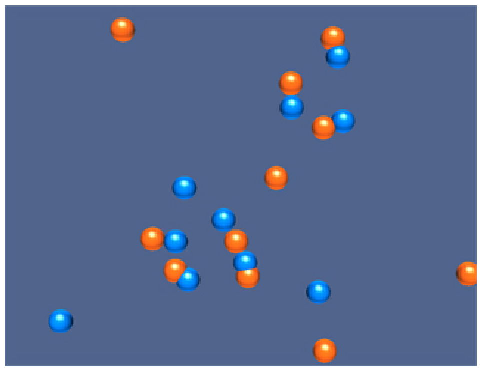


Figure 5.8.3 A two dimensional interactive simulation of a collection of positive and negative charges affected by the Coulomb force and the Pauli repulsive force, with dynamic damping.

In particular, try this sequence of actions with the display. Start it and wait until the simulation has evolved to the point where you have a line of particles made up of seven or eight particles. Left click on one of the end charges of this line and drag it with the mouse. If you do this slowly enough, the entire line of chares will follow along with the charge you are virtually "touching". When you move that charge, you are putting "energy" into the charge you have selected on one end of the line. This "energy" is going into moving that charge, but it is also being supplied to the rest of the charges via their electromagnetic fields. The "energy" that the charge on the opposite end of the line receives a little while after you start moving the first charge is delivered to it entirely by energy flowing through space in the electromagnetic field, from the site where you create that energy.

This is a microcosm of how you interact with the world. A physical object lying on the floor in front is held together by electrostatic forces. Quantum mechanics keeps it from collapsing; electrostatic forces keep it from flying apart. When you reach down and pick that object up by one end, energy is transferred from where you grasp the object to the rest of it by energy flow in the electromagnetic field. When you raise it above the floor, the "tail end" of the object never "touches" the point where you grasp it. All of the energy provided to the "tail end" of the object to move it upward against gravity is provided by energy flow via electromagnetic fields, through the complicated web of electromagnetic fields that hold the object together.

Interactive Simulation 5.4: **Collection of Charges in Three Dimensions**

Figure 5.8.4 is an interactive three-dimensional ShockWave display that shows the same dynamical situation as in Figure 5.8.3 except that we are looking at the scene in three dimensions. This display can be rotated to view from different angles by right-clicking and dragging in the display. We start the charges at rest in random positions in space, and then let them evolve according to the forces that act on them (electrostatic attraction/repulsion, Pauli repulsion at very short distances, and a dynamic drag term proportional to velocity). Here the configurations are more complex because of the availability of the third dimension. In particular, one can hit the "w" key to toggle a force that pushes the charges together on and off. Toggling this force *on* and letting the charges settle down in a "clump", and then toggling it *off* to let them expand, allows the construction of complicated three dimension structures that are "meta-stable". An example of one of these is given in Figure 5.8.4.

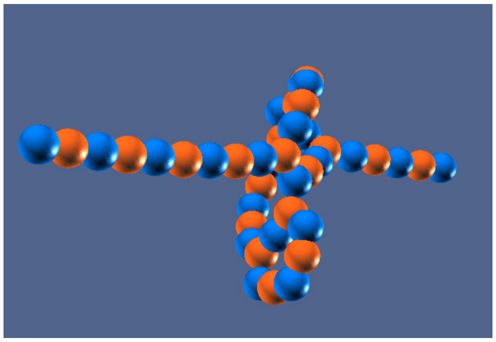


Figure 5.8.4 An three-dimensional interactive simulation of a collection of positive and negative charges affected by the Coulomb force and the Pauli repulsive force, with dynamic damping.

Interactive Simulation 5.5: **Collection of Dipoles in Two Dimensions**

Figure 5.8.5 shows an interactive ShockWave simulation that allows one to interact in two dimensions with a group of electric dipoles.

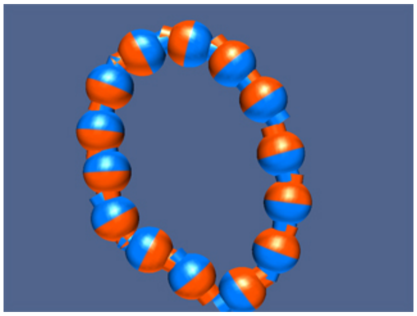


Figure 5.8.5 An interactive simulation of a collection of electric dipoles affected by the Coulomb force and the Pauli repulsive force, with dynamic damping.

The dipoles are created with random positions and orientations, with all the electric dipole vectors in the plane of the display. As we noted above, although in principle the dipole-dipole interaction can be either repulsive or attractive, in practice there is a torque that rotates the dipoles so that the dipole-dipole force is attractive. In the ShockWave simulation we see this behavior—that is, the dipoles orient themselves so as to attract, and then the attraction gathers them together into bound structures.

Interactive Simulation 5.6: Charged Particle Trap

Figure 5.8.6 shows an interactive simulation of a charged particle trap.

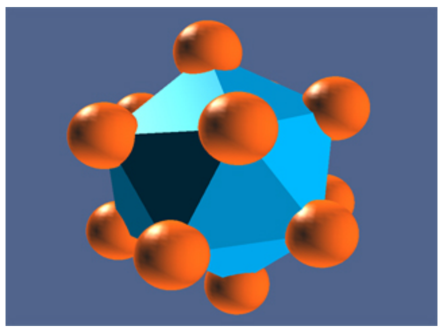


Figure 5.8.6 An interactive simulation of a particle trap.

Particles interact as before, but in addition each particle feels a force that pushes them toward the origin, regardless of the sign of their charge. That "trapping" force increases linearly with distance from the origin. The charges initially are randomly distributed in space, but as time increases the dynamic damping "cools" the particles and they "crystallize" into a number of highly symmetric structures, depending on the number of particles. This mimics the highly ordered structures that we see in nature (e.g., snowflakes).

Exercise:

Start the simulation. The simulation initially introduces 12 positive charges in random positions (you can of course add more particles of either sign, but for the moment we deal with only the initial 12). About half the time, the 12 charges will settle down into an equilibrium in which there is a charge in the center of a sphere on which the other 11 charges are arranged. The other half of the time all 12 particles will be arranged on the surface of a sphere, with no charge in the middle. Whichever arrangement you initially find, see if you can move one of the particles into position so that you get to the other stable configuration. To move a charge, push shift and left click, and use the arrow buttons to move it up, down, left, and right. You may have to select several different charges in turn to find one that you can move into the center, if you initial equilibrium does not have a center charge.

Here is another exercise. Put an additional 8 positive charges into the display (by pressing "p" eight times) for a total of 20 charges. By moving charges around as above, you can get two charges in inside a spherical distribution of the other 18. Is this the lowest number of charges for which you can get equilibrium with two charges inside? That is, can you do this with 18 charges? Note that if you push the "s" key you will get generate a surface based on the positions of the charges in the sphere, which will make its symmetries more apparent.

Interactive Simulation 5.6: Lattice 3D

Lattice 3D, shown in Figure 5.8.7, simulates the interaction of charged particles in three dimensions. The particles interact via the classical Coulomb force, as well as the repulsive quantum-mechanical Pauli force, which acts at close distances (accounting for the "collisions" between them). Additionally, the motion of the particles is damped by a term proportional to their velocity, allowing them to "settle down" into stable (or meta-stable) states.

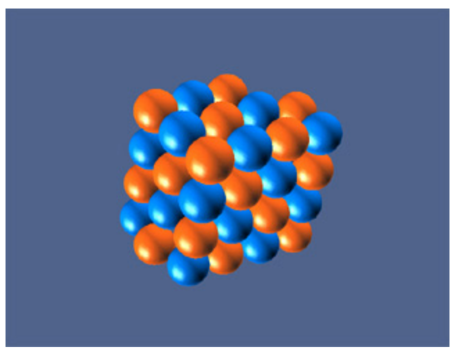


Figure 5.8.7 *Lattice 3D* simulating the interaction of charged particles in three dimensions.

In this simulation, the proportionality of the Coulomb and Pauli forces has been adjusted to allow for lattice formation, as one might see in a crystal. The "preferred" stable state is a rectangular (cubic) lattice, although other formations are possible depending on the number of particles and their initial positions.

Selecting a particle and pressing "f" will toggle field lines illustrating the local field around that particle. Performance varies depending on the number of particles / field lines in the simulation.

Interactive Simulation 5.7: 2D Electrostatic Suspension Bridge

To connect electrostatic forces to one more example of the real world, Figure 5.8.8 is a simulation of a 2D "electrostatic suspension bridge." The bridge is created by attaching a series of positive and negatively charged particles to two fixed endpoints, and adding a downward gravitational force. The tension in the "bridge" is supplied simply by the

Coulomb interaction of its constituent parts and the Pauli force keeps the charges from collapsing in on each other. Initially, the bridge only sags slightly under the weight of gravity. However the user can introduce additional "neutral" particles (by pressing "o") to stress the bridge more, until the electrostatic bonds "break" under the stress and the bridge collapses.

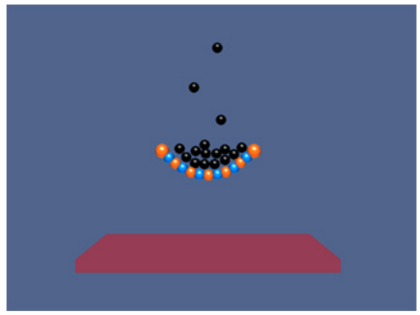


Figure 5.8.8 A ShockWave simulation of a 2D electrostatic suspension bridge.

Interactive Simulation 5.8: 3D Electrostatic Suspension Bridge

In the simulation shown in Figure 5.8.9, a 3D "electostatic suspension bridge" is created by attaching a lattice of positive and negatively charged particles between four fixed corners, and adding a downward gravitational force. The tension in the "bridge" is supplied simply by the Coulomb interaction of its constituent parts and the Pauli force keeping them from collapsing in on each other. Initially, the bridge only sags slightly under the weight of gravity, but what would happen to it under a rain of massive neutral particles? Press "o" to find out.

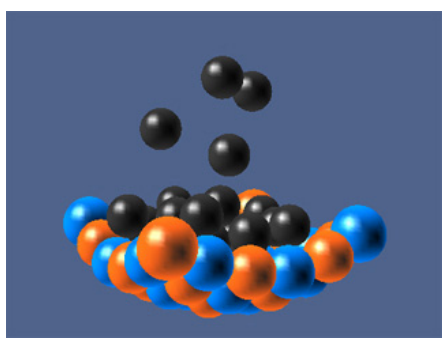


Figure 5.8.9 A ShockWave simulation of a 3D electrostatic suspension bridge.

5.9 Problem-Solving Strategy: Calculating Capacitance

In this chapter, we have seen how capacitance C can be calculated for various systems. The procedure is summarized below:

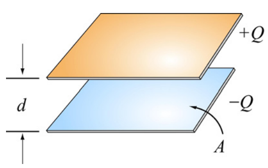
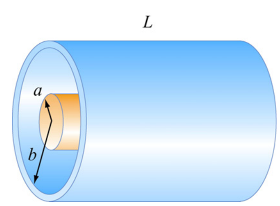
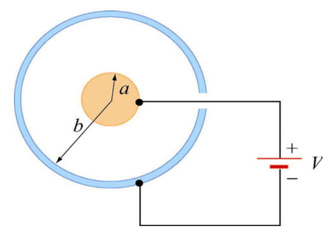
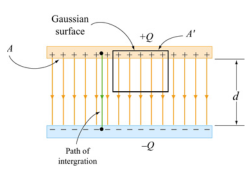
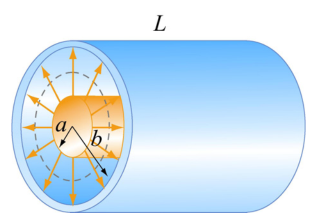
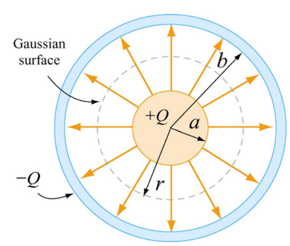
(1) Identify the direction of the electric field using symmetry.

(2) Calculate the electric field everywhere.

(3) Compute the electric potential difference ΔV.

(4) Calculate the capacitance C using $C = Q / |\Delta V|$.

In the Table below, we illustrate how the above steps are used to calculate the capacitance of a parallel-plate capacitor, cylindrical capacitor and a spherical capacitor.

Capacitors	**Parallel-plate**	**Cylindrical**	**Spherical**
Figure			
(1) Identify the direction of the electric field using symmetry			
(2) Calculate electric field everywhere	$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = EA = \dfrac{Q}{\varepsilon_0}$ $E = \dfrac{Q}{A\varepsilon_0} = \dfrac{\sigma}{\varepsilon_0}$	$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E(2\pi r l) = \dfrac{Q}{\varepsilon_0}$ $E = \dfrac{\lambda}{2\pi\varepsilon_0 r}$	$\oiint_S \vec{\mathbf{E}} \cdot d\vec{\mathbf{A}} = E_r\left(4\pi r^2\right) = \dfrac{Q}{\varepsilon_0}$ $E_r = \dfrac{1}{4\pi\varepsilon_o}\dfrac{Q}{r^2}$
(3) Compute the electric potential difference ΔV	$\Delta V = V_- - V_+ = -\int_+^- \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$ $= -Ed$	$\Delta V = V_b - V_a = -\int_a^b E_r dr$ $= -\dfrac{\lambda}{2\pi\varepsilon_0}\ln\left(\dfrac{b}{a}\right)$	$\Delta V = V_b - V_a = -\int_a^b E_r dr$ $= -\dfrac{Q}{4\pi\varepsilon_0}\left(\dfrac{b-a}{ab}\right)$

(4) Calculate C using $C = Q/\lvert\Delta V\rvert$	$C = \dfrac{\varepsilon_0 A}{d}$	$C = \dfrac{2\pi\varepsilon_0 l}{\ln(b/a)}$	$C = 4\pi\varepsilon_0\left(\dfrac{ab}{b-a}\right)$

5.10 Solved Problems

5.10.1 Equivalent Capacitance

Consider the configuration shown in Figure 5.10.1. Find the equivalent capacitance, assuming that all the capacitors have the same capacitance C.

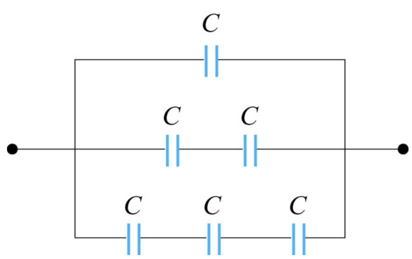


Figure 5.10.1 Combination of Capacitors

Solution:

For capacitors that are connected in series, the equivalent capacitance is

$$\frac{1}{C_{\text{eq}}} = \frac{1}{C_1} + \frac{1}{C_2} + \cdots = \sum_i \frac{1}{C_i} \qquad \text{(series)}$$

On the other hand, for capacitors that are connected in parallel, the equivalent capacitance is

$$C_{\text{eq}} = C_1 + C_2 + \cdots = \sum_i C_i \qquad \text{(parallel)}$$

Using the above formula for series connection, the equivalent configuration is shown in Figure 5.10.2.

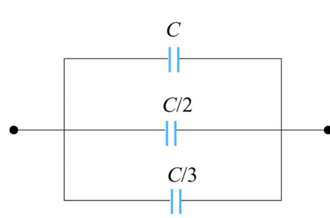


Figure 5.10.2

Now we have three capacitors connected in parallel. The equivalent capacitance is given by

$$C_{\text{eq}} = C\left(1+\frac{1}{2}+\frac{1}{3}\right) = \frac{11}{6}C$$

5.10.2 Capacitor Filled with Two Different Dielectrics

Two dielectrics with dielectric constants κ_1 and κ_2 each fill half the space between the plates of a parallel-plate capacitor as shown in Figure 5.10.3.

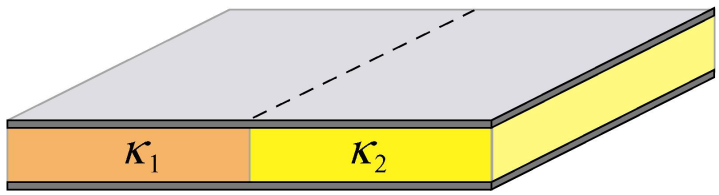


Figure 5.10.3 Capacitor filled with two different dielectrics.

Each plate has an area A and the plates are separated by a distance d. Compute the capacitance of the system.

Solution:

Since the potential difference on each half of the capacitor is the same, we may treat the system as being composed of two capacitors connected in parallel. Thus, the capacitance of the system is

$$C = C_1 + C_2$$

With

$$C_i = \frac{\kappa_i \varepsilon_0 (A/2)}{d}, \quad i = 1,2$$

we obtain

$$C = \frac{\kappa_1 \varepsilon_0 (A/2)}{d} + \frac{\kappa_2 \varepsilon_0 (A/2)}{d} = \frac{\varepsilon_0 A}{2d}(\kappa_1 + \kappa_2)$$

5.10.3 Capacitor with Dielectrics

Consider a conducting spherical shell with an inner radius a and outer radius c. Let the space between two surfaces be filed with two different dielectric materials so that the dielectric constant is κ_1 between a and b, and κ_2 between b and c, as shown in Figure 5.10.4. Determine the capacitance of this system.

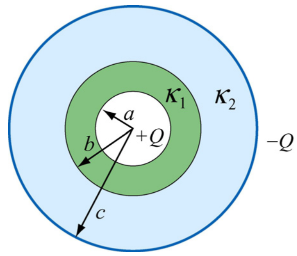


Figure 5.10.4 Spherical capacitor filled with dielectrics.

Solution:

The system can be treated as two capacitors connected in series, since the total potential difference across the capacitors is the sum of potential differences across individual capacitors. The equivalent capacitance for a spherical capacitor of inner radius r_1 and outer radius r_2 filled with dielectric with dielectric constant κ_e is given by

$$C = 4\pi\varepsilon_0\kappa_e\left(\frac{r_1 r_2}{r_2 - r_1}\right)$$

Thus, the equivalent capacitance of this system is

$$\frac{1}{C} = \frac{1}{\dfrac{4\pi\varepsilon_0\kappa_1 ab}{(b-a)}} + \frac{1}{\dfrac{4\pi\varepsilon_0\kappa_2 bc}{(c-b)}} = \frac{\kappa_2 c(b-a) + \kappa_1 a(c-b)}{4\pi\varepsilon_0\kappa_1\kappa_2 abc}$$

or

$$\boxed{C = \frac{4\pi\varepsilon_0\kappa_1\kappa_2 abc}{\kappa_2 c(b-a) + \kappa_1 a(c-b)}}$$

It is instructive to check the limit where $\kappa_1, \kappa_2 \to 1$. In this case, the above expression reduces to

$$C = \frac{4\pi\varepsilon_0 abc}{c(b-a) + a(c-b)} = \frac{4\pi\varepsilon_0 abc}{b(c-a)} = \frac{4\pi\varepsilon_0 ac}{(c-a)}$$

which agrees with Eq. (5.2.11) for a spherical capacitor of inner radius a and outer radius c.

5.10.4 Capacitor Connected to a Spring

Consider an air-filled parallel-plate capacitor with one plate connected to a spring having a force constant *k*, and another plate held fixed. The system rests on a table top as shown in Figure 5.10.5.

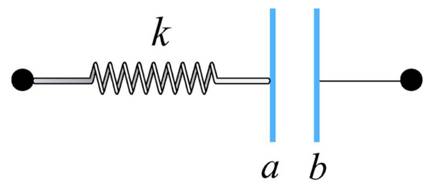


Figure 5.10.5 Capacitor connected to a spring.

If the charges placed on plates *a* and *b* are $+Q$ and $-Q$, respectively, how much does the spring expand?

Solution:

The spring force $\vec{\mathbf{F}}_s$ acting on plate *a* is given by

$$\vec{\mathbf{F}}_s = -kx\,\hat{\mathbf{i}}$$

Similarly, the electrostatic force $\vec{\mathbf{F}}_e$ due to the electric field created by plate *b* is

$$\vec{\mathbf{F}}_e = QE\,\hat{\mathbf{i}} = Q\left(\frac{\sigma}{2\varepsilon_0}\right)\hat{\mathbf{i}} = \frac{Q^2}{2A\varepsilon_0}\hat{\mathbf{i}}$$

where A is the area of the plate . Notice that charges on plate *a* cannot exert a force on itself, as required by Newton's third law. Thus, only the electric field due to plate *b* is considered. At equilibrium the two forces cancel and we have

$$kx = Q\left(\frac{Q}{2A\varepsilon_0}\right)$$

which gives

$$x = \frac{Q^2}{2kA\varepsilon_0}$$

5.11 Conceptual Questions

1. The charges on the plates of a parallel-plate capacitor are of opposite sign, and they attract each other. To increase the plate separation, is the external work done positive or negative? What happens to the external work done in this process?

2. How does the stored energy change if the potential difference across a capacitor is tripled?

3. Does the presence of a dielectric increase or decrease the maximum operating voltage of a capacitor? Explain.

4. If a dielectric-filled capacitor is cooled down, what happens to its capacitance?

5.12 Additional Problems

5.12.1 Capacitors in Series and in Parallel

A 12-Volt battery charges the four capacitors shown in Figure 5.12.1.

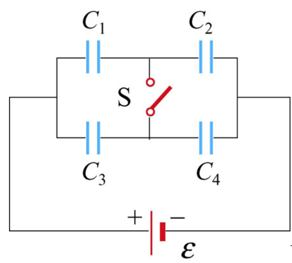


Figure 5.12.1

Let $C_1 = 1\ \mu\text{F}$, $C_2 = 2\ \mu\text{F}$, $C_3 = 3\ \mu\text{F}$, and $C_4 = 4\ \mu\text{F}$.

(a) What is the equivalent capacitance of the group C_1 and C_2 if switch S is open (as shown)?

(b) What is the charge on *each* of the four capacitors if switch S is open?

(c) What is the charge on each of the four capacitors if switch S is closed?

5.12.2 Capacitors and Dielectrics

(a) A parallel-plate capacitor of area A and spacing d is filled with three dielectrics as shown in Figure 5.12.2. Each occupies 1/3 of the volume. What is the capacitance of this system? [*Hint:* Consider an equivalent system to be three parallel capacitors, and justify this assumption.] Show that you obtain the proper limits as the dielectric constants approach unity, $\kappa_i \rightarrow 1$.]

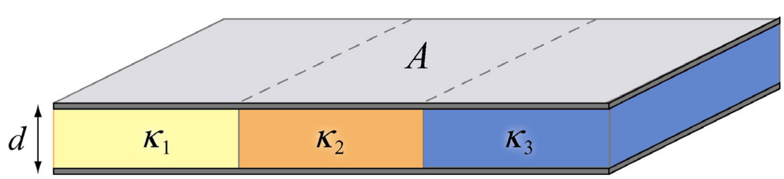


Figure 5.12.2

(b) This capacitor is now filled as shown in Figure 5.12.3. What is its capacitance? Use Gauss's law to find the field in each dielectric, and then calculate ΔV across the entire capacitor. Again, check your answer as the dielectric constants approach unity, $\kappa_i \to 1$. Could you have assumed that this system is equivalent to three capacitors in series?

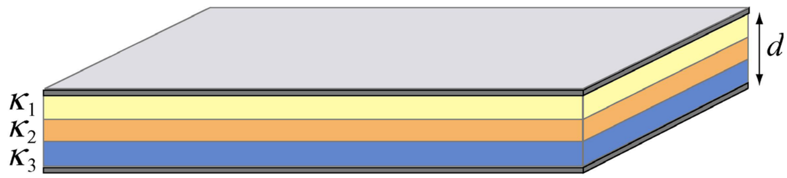


Figure 5.12.3

5.12.3 Gauss's Law in the Presence of a Dielectric

A solid conducting sphere with a radius R_1 carries a free charge Q and is surrounded by a concentric dielectric spherical shell with an outer radius R_2 and a dielectric constant κ_e. This system is isolated from other conductors and resides in air ($\kappa_e \approx 1$), as shown in Figure 5.12.4.

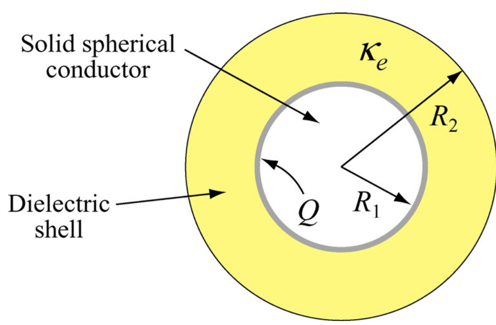


Figure 5.12.4

(a) Determine the displacement vector $\vec{\mathbf{D}}$ everywhere, *i.e.* its magnitude and direction in the regions $r < R_1$, $R_1 < r < R_2$ and $r > R_2$.

(b) Determine the electric field $\vec{\mathbf{E}}$ everywhere.

5.12.4 Gauss's Law and Dielectrics

A cylindrical shell of dielectric material has inner radius a and outer radius b, as shown in Figure 5.12.5.

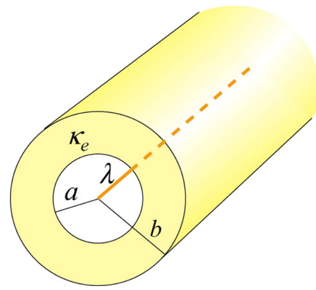


Figure 5.12.5

The material has a dielectric constant $\kappa_e = 10$. At the center of the shell there is a line charge running parallel to the axis of the cylindrical shell, with free charge per unit length λ.

(a) Find the electric field for: $r < a$, $a < r < b$ and $r > b$.

(b) What is the induced surface charge per unit length on the inner surface of the spherical shell? [Ans: $-9\lambda/10$.]

(c) What is the induced surface charge per unit length on the outer surface of the spherical shell? [Ans: $+9\lambda/10$.]

5.12.5 A Capacitor with a Dielectric

A parallel plate capacitor has a capacitance of 112 pF, a plate area of 96.5 cm^2, and a mica dielectric ($\kappa_e = 5.40$). At a 55 V potential difference, calculate

(a) the electric field strength in the mica; [Ans: 13.4 kV/m.]

(b) the magnitude of the free charge on the plates; [Ans: 6.16 nC.]

(c) the magnitude of the induced surface charge; [Ans: 5.02 nC.]

(d) the magnitude of the polarization $\vec{\mathbf{P}}$ [Ans: 520 nC/m^2.]

5.12.6 Force on the Plates of a Capacitor

The plates of a parallel-plate capacitor have area A and carry total charge $\pm Q$ (see Figure 5.12.6). We would like to show that these plates *attract* each other with a force given by $F = Q^2/(2\varepsilon_0 A)$.

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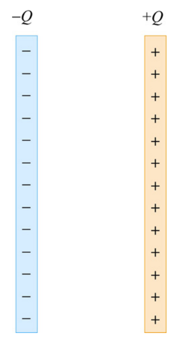


Figure 5.12.6

(a) Calculate the total force on the left plate due to the electric field of the right plate, using Coulomb's Law. Ignore fringing fields.

(b) If you pull the plates apart, against their attraction, you are doing work and *that work goes directly into creating additional electrostatic energy*. Calculate the force necessary to increase the plate separation from x to $x+dx$ by equating the work you do, $\vec{\mathbf{F}}\cdot d\vec{\mathbf{x}}$, to the increase in electrostatic energy, assuming that the electric energy density is $\varepsilon_o E^2/2$, and that the charge Q remains constant.

(c) Using this expression for the force, show that the force per unit area (the *electrostatic stress*) acting on either capacitor plate is given by $\varepsilon_o E^2/2$. This result is true for a conductor of any shape with an electric field $\vec{\mathbf{E}}$ at its surface.

(d) Atmospheric pressure is 14.7 lb/in^2, or 101,341 N/m^2. How large would E have to be to produce this force per unit area? [Ans: 151 MV/m. Note that Van de Graff accelerators can reach fields of 100 MV/m maximum before breakdown, so that electrostatic stresses are on the same order as atmospheric pressures in this extreme situation, but not much greater].

5.12.7 Energy Density in a Capacitor with a Dielectric

Consider the case in which a dielectric material with dielectric constant κ_e completely fills the space between the plates of a parallel-plate capacitor. Show that the energy density of the field between the plates is $u_E = \vec{\mathbf{E}}\cdot\vec{\mathbf{D}}/2$ by the following procedure:

(a) Write the expression $u_E = \vec{\mathbf{E}}\cdot\vec{\mathbf{D}}/2$ as a function of $\mathbf{E}$ and κ_e (i.e. eliminate $\vec{\mathbf{D}}$).

(b) Given the electric field and potential of such a capacitor with free charge q on it (problem 4-1a above), calculate the work done to charge up the capacitor from $q=0$ to $q=Q$, the final charge.

(c) Find the energy density u_E.

PHYSICS

Study material of Class XII Science Stream

Chapter 3

Electric Potential

Electric Potential

3.1 Potential and Potential Energy

In the introductory mechanics course, we have seen that gravitational force from the Earth on a particle of mass *m* located at a distance *r* from Earth's center has an inverse-square form:

$$\vec{\mathbf{F}}_g = -G\frac{Mm}{r^2}\hat{\mathbf{r}} \tag{3.1.1}$$

where $G = 6.67\times10^{-11}\ \mathrm{N\cdot m^2/kg^2}$ is the gravitational constant and $\hat{\mathbf{r}}$ is a unit vector pointing radially outward. The Earth is assumed to be a uniform sphere of mass *M*. The corresponding gravitational field $\vec{\mathbf{g}}$, defined as the gravitational force per unit mass, is given by

$$\vec{\mathbf{g}} = \frac{\vec{\mathbf{F}}_g}{m} = -\frac{GM}{r^2}\hat{\mathbf{r}} \tag{3.1.2}$$

Notice that $\vec{\mathbf{g}}$ only depends on *M*, the mass which creates the field, and *r*, the distance from *M*.

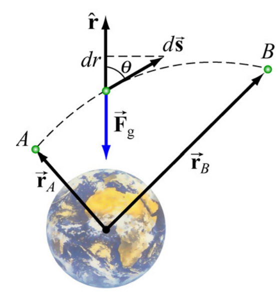


Figure 3.1.1

Consider moving a particle of mass m under the influence of gravity (Figure 3.1.1). The work done by gravity in moving m from *A* to *B* is

$$W_g = \int \vec{\mathbf{F}}_g \cdot d\,\vec{\mathbf{s}} = \int_{r_A}^{r_B}\left(-\frac{GMm}{r^2}\right)dr = \left[\frac{GMm}{r}\right]_{r_A}^{r_B} = GMm\left(\frac{1}{r_B}-\frac{1}{r_A}\right) \tag{3.1.3}$$

The result shows that W_g is independent of the path taken; it depends only on the endpoints *A* and *B*. It is important to draw distinction between W_g, the work done by the field and W_{ext}, the work done by an external agent such as you. They simply differ by a negative sign: $W_g = -W_{\text{ext}}$.

Near Earth's surface, the gravitational field $\vec{\mathbf{g}}$ is approximately constant, with a magnitude $g = GM/r_E^2 \approx 9.8\,\text{m/s}^2$, where r_E is the radius of Earth. The work done by gravity in moving an object from height y_A to y_B (Figure 3.1.2) is

$$W_g = \int \vec{\mathbf{F}}_g \cdot d\vec{\mathbf{s}} = \int_A^B mg\cos\theta\, ds = -\int_A^B mg\cos\phi\, ds = -\int_{y_A}^{y_B} mg\, dy = -mg(y_B - y_A) \qquad (3.1.4)$$

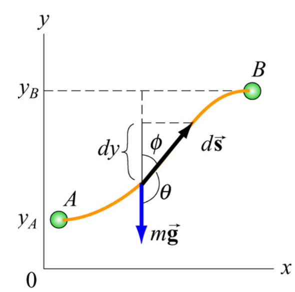


Figure 3.1.2 Moving a mass *m* from *A* to *B*.

The result again is independent of the path, and is only a function of the change in vertical height $y_B - y_A$.

In the examples above, if the path forms a closed loop, so that the object moves around and then returns to where it starts off, the net work done by the gravitational field would be zero, and we say that the gravitational force is conservative. More generally, a force $\vec{\mathbf{F}}$ is said to be *conservative* if its line integral around a closed loop vanishes:

$$\boxed{\oint \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = 0} \qquad (3.1.5)$$

When dealing with a conservative force, it is often convenient to introduce the concept of potential energy *U*. The change in potential energy associated with a conservative force $\vec{\mathbf{F}}$ acting on an object as it moves from *A* to *B* is defined as:

$$\boxed{\Delta U = U_B - U_A = -\int_A^B \vec{\mathbf{F}} \cdot d\vec{\mathbf{s}} = -W} \qquad (3.1.6)$$

where W is the work done by the force on the object. In the case of gravity, $W = W_g$ and from Eq. (3.1.3), the potential energy can be written as

$$U_g = -\frac{GMm}{r} + U_0 \qquad (3.1.7)$$

where U_0 is an arbitrary constant which depends on a reference point. It is often convenient to choose a reference point where U_0 is equal to zero. In the gravitational case, we choose infinity to be the reference point, with $U_0(r=\infty)=0$. Since U_g depends on the reference point chosen, it is only the potential energy difference ΔU_g that has physical importance. Near Earth's surface where the gravitational field $\vec{\mathbf{g}}$ is approximately constant, as an object moves from the ground to a height *h*, the change in potential energy is $\Delta U_g = +mgh$, and the work done by gravity is $W_g = -mgh$.

A concept which is closely related to potential energy is "potential." From ΔU, the gravitational potential can be obtained as

$$\boxed{\Delta V_g = \frac{\Delta U_g}{m} = -\int_A^B (\vec{\mathbf{F}}_g / m)\cdot d\,\vec{\mathbf{s}} = -\int_A^B \vec{\mathbf{g}}\cdot d\,\vec{\mathbf{s}}} \tag{3.1.8}$$

Physically ΔV_g represents the negative of the work done per unit mass by gravity to move a particle from A to B.

Our treatment of electrostatics is remarkably similar to gravitation. The electrostatic force $\vec{\mathbf{F}}_e$ given by Coulomb's law also has an inverse-square form. In addition, it is also conservative. In the presence of an electric field $\vec{\mathbf{E}}$, in analogy to the gravitational field $\vec{\mathbf{g}}$, we define the electric potential difference between two points A and B as

$$\boxed{\Delta V = -\int_A^B (\vec{\mathbf{F}}_e / q_0)\cdot d\,\vec{\mathbf{s}} = -\int_A^B \vec{\mathbf{E}}\cdot d\,\vec{\mathbf{s}}} \tag{3.1.9}$$

where q_0 is a test charge. The potential difference ΔV represents the amount of work done per unit charge to move a test charge q_0 from point *A* to *B*, without changing its kinetic energy. Again, electric potential should not be confused with electric potential energy. The two quantities are related by

$$\Delta U = q_0 \Delta V \tag{3.1.10}$$

The SI unit of electric potential is volt (V):

$$1\,\text{volt} = 1\ \text{joule/coulomb}\ \ (1\ \text{V} = 1\ \text{J/C}) \tag{3.1.11}$$

When dealing with systems at the atomic or molecular scale, a joule (J) often turns out to be too large as an energy unit. A more useful scale is electron volt (eV), which is defined as the energy an electron acquires (or loses) when moving through a potential difference of one volt:

$$1\,\text{eV} = (1.6\times10^{-19}\,\text{C})(1\,\text{V}) = 1.6\times10^{-19}\,\text{J} \tag{3.1.12}$$

3.2 Electric Potential in a Uniform Field

Consider a charge $+q$ moving in the direction of a uniform electric field $\vec{\mathbf{E}} = E_0(-\hat{\mathbf{j}})$, as shown in Figure 3.2.1(a).

Figure 3.2.1 (a) A charge q which moves in the direction of a constant electric field $\vec{\mathbf{E}}$. (b) A mass m that moves in the direction of a constant gravitational field $\vec{\mathbf{g}}$.

Since the path taken is parallel to $\vec{\mathbf{E}}$, the potential difference between points A and B is given by

$$\Delta V = V_B - V_A = -\int_A^B \vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} = -E_0\int_A^B ds = -E_0 d < 0 \tag{3.2.1}$$

implying that point B is at a lower potential compared to A. In fact, electric field lines always point from higher potential to lower. The change in potential energy is $\Delta U = U_B - U_A = -qE_0 d$. Since $q > 0$, we have $\Delta U < 0$, which implies that the potential energy of a positive charge decreases as it moves along the direction of the electric field. The corresponding gravitational analogy, depicted in Figure 3.2.1(b), is that a mass m loses potential energy ($\Delta U = -mgd$) as it moves in the direction of the gravitational field $\vec{\mathbf{g}}$.

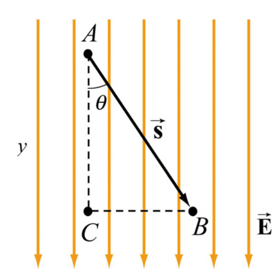


Figure 3.2.2 Potential difference due to a uniform electric field

What happens if the path from A to B is not parallel to $\vec{\mathbf{E}}$, but instead at an angle θ, as shown in Figure 3.2.2? In that case, the potential difference becomes

$$\Delta V = V_B - V_A = -\int_A^B \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} = -\vec{\mathbf{E}} \cdot \vec{\mathbf{s}} = -E_0 s \cos\theta = -E_0 y \qquad (3.2.2)$$

Note that y increase downward in Figure 3.2.2. Here we see once more that moving along the direction of the electric field $\vec{\mathbf{E}}$ leads to a lower electric potential. What would the change in potential be if the path were $A \to C \to B$? In this case, the potential difference consists of two contributions, one for each segment of the path:

$$\Delta V = \Delta V_{CA} + \Delta V_{BC} \qquad (3.2.3)$$

When moving from A to C, the change in potential is $\Delta V_{CA} = -E_0 y$. On the other hand, when going from C to B, $\Delta V_{BC} = 0$ since the path is perpendicular to the direction of $\vec{\mathbf{E}}$. Thus, the same result is obtained irrespective of the path taken, consistent with the fact that $\vec{\mathbf{E}}$ is conservative.

Notice that for the path $A \to C \to B$, work is done by the field only along the segment AC which is parallel to the field lines. Points B and C are at the same electric potential, i.e., $V_B = V_C$. Since $\Delta U = q\Delta V$, this means that no work is required in moving a charge from B to C. In fact, all points along the straight line connecting B and C are on the same "equipotential line." A more complete discussion of equipotential will be given in Section 3.5.

3.3 Electric Potential due to Point Charges

Next, let's compute the potential difference between two points A and B due to a charge $+Q$. The electric field produced by Q is $\vec{\mathbf{E}} = (Q / 4\pi\varepsilon_0 r^2)\hat{\mathbf{r}}$, where $\hat{\mathbf{r}}$ is a unit vector pointing toward the field point.

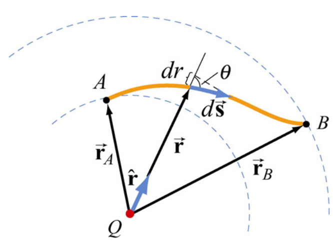


Figure 3.3.1 Potential difference between two points due to a point charge Q.

From Figure 3.3.1, we see that $\hat{\mathbf{r}} \cdot d\vec{\mathbf{s}} = ds\cos\theta = dr$, which gives

$$\Delta V = V_B - V_A = -\int_A^B \frac{Q}{4\pi\varepsilon_0 r^2} \hat{\mathbf{r}} \cdot d\,\vec{\mathbf{s}} = -\int_A^B \frac{Q}{4\pi\varepsilon_0 r^2} dr = \frac{Q}{4\pi\varepsilon_0}\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \qquad (3.3.1)$$

Once again, the potential difference ΔV depends only on the endpoints, independent of the choice of path taken.

As in the case of gravity, only the difference in electrical potential is physically meaningful, and one may choose a reference point and set the potential there to be zero. In practice, it is often convenient to choose the reference point to be at infinity, so that the electric potential at a point P becomes

$$V_P = -\int_{\infty}^{P} \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}} \tag{3.3.2}$$

With this reference, the electric potential at a distance r away from a point charge Q becomes

$$\boxed{V(r) = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}} \tag{3.3.3}$$

When more than one point charge is present, by applying the superposition principle, the total electric potential is simply the sum of potentials due to individual charges:

$$V(r) = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{q_i}{r_i} = k_e \sum_i \frac{q_i}{r_i} \tag{3.3.4}$$

A summary of comparison between gravitation and electrostatics is tabulated below:

Gravitation	Electrostatics
Mass m	Charge q
Gravitational force $\vec{\mathbf{F}}_g = -G\dfrac{Mm}{r^2}\hat{\mathbf{r}}$	Coulomb force $\vec{\mathbf{F}}_e = k_e\dfrac{Qq}{r^2}\hat{\mathbf{r}}$
Gravitational field $\vec{\mathbf{g}} = \vec{\mathbf{F}}_g / m$	Electric field $\vec{\mathbf{E}} = \vec{\mathbf{F}}_e / q$
Potential energy change $\Delta U = -\int_A^B \vec{\mathbf{F}}_g \cdot d\,\vec{\mathbf{s}}$	Potential energy change $\Delta U = -\int_A^B \vec{\mathbf{F}}_e \cdot d\,\vec{\mathbf{s}}$
Gravitational potential $V_g = -\int_A^B \vec{\mathbf{g}} \cdot d\,\vec{\mathbf{s}}$	Electric Potential $V = -\int_A^B \vec{\mathbf{E}} \cdot d\,\vec{\mathbf{s}}$
For a source M: $V_g = -\dfrac{GM}{r}$	For a source Q: $V = k_e\dfrac{Q}{r}$
$\lvert \Delta U_g \rvert = mgd$ (constant $\vec{\mathbf{g}}$)	$\lvert \Delta U \rvert = qEd$ (constant $\vec{\mathbf{E}}$)

3.3.1 Potential Energy in a System of Charges

If a system of charges is assembled by an external agent, then $\Delta U = -W = +W_{\text{ext}}$. That is, the change in potential energy of the system is the work that must be put in by an external agent to assemble the configuration. A simple example is lifting a mass m through a height h. The work done by an external agent ˜ you, is $+mgh$ (The gravitational field does work $-mgh$). The charges are brought in from infinity without acceleration i.e. they are at rest at the end of the process. Let's start with just two charges q_1 and q_2. Let the potential due to q_1 at a point P be V_1 (Figure 3.3.2).

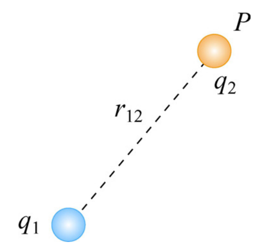


Figure 3.3.2 Two point charges separated by a distance r_{12}.

The work W_2 done by an agent in bringing the second charge q_2 from infinity to P is then $W_2 = q_2 V_1$. (No work is required to set up the first charge and $W_1 = 0$). Since $V_1 = q_1 / 4\pi\varepsilon_0 r_{12}$, where r_{12} is the distance measured from q_1 to P, we have

$$U_{12} = W_2 = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}} \tag{3.3.5}$$

If q_1 and q_2 have the same sign, positive work must be done to overcome the electrostatic repulsion and the potential energy of the system is positive, $U_{12} > 0$. On the other hand, if the signs are opposite, then $U_{12} < 0$ due to the attractive force between the charges.

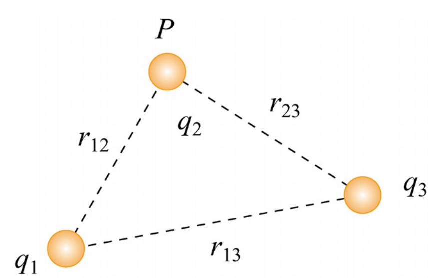


Figure 3.3.3 A system of three point charges.

To add a third charge q_3 to the system (Figure 3.3.3), the work required is

$$W_3 = q_3\left(V_1+V_2\right)=\frac{q_3}{4\pi\varepsilon_0}\left(\frac{q_1}{r_{13}}+\frac{q_2}{r_{23}}\right) \tag{3.3.6}$$

The potential energy of this configuration is then

$$U = W_2 + W_3 = \frac{1}{4\pi\varepsilon_0}\left(\frac{q_1 q_2}{r_{12}}+\frac{q_1 q_3}{r_{13}}+\frac{q_2 q_3}{r_{23}}\right)=U_{12}+U_{13}+U_{23} \tag{3.3.7}$$

The equation shows that the total potential energy is simply the sum of the contributions from distinct pairs. Generalizing to a system of N charges, we have

$$U = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j>i}}^{N}\frac{q_i q_j}{r_{ij}} \tag{3.3.8}$$

where the constraint $j > i$ is placed to avoid double counting each pair. Alternatively, one may count each pair twice and divide the result by 2. This leads to

$$U = \frac{1}{8\pi\varepsilon_0}\sum_{i=1}^{N}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_i q_j}{r_{ij}} = \frac{1}{2}\sum_{i=1}^{N} q_i\left(\frac{1}{4\pi\varepsilon_0}\sum_{\substack{j=1\\ j\neq i}}^{N}\frac{q_j}{r_{ij}}\right) = \frac{1}{2}\sum_{i=1}^{N} q_i V(r_i) \tag{3.3.9}$$

where $V(r_i)$, the quantity in the parenthesis, is the potential at $\vec{\mathbf{r}}_i$ (location of q_i) due to all the other charges.

3.4 Continuous Charge Distribution

If the charge distribution is continuous, the potential at a point P can be found by summing over the contributions from individual differential elements of charge dq.

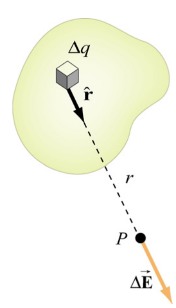


Figure 3.4.1 Continuous charge distribution

Consider the charge distribution shown in Figure 3.4.1. Taking infinity as our reference point with zero potential, the electric potential at P due to dq is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} \tag{3.4.1}$$

Summing over contributions from all differential elements, we have

$$\boxed{V = \frac{1}{4\pi\varepsilon_0}\int\frac{dq}{r}} \tag{3.4.2}$$

3.5 Deriving Electric Field from the Electric Potential

In Eq. (3.1.9) we established the relation between $\vec{\mathbf{E}}$ and V. If we consider two points which are separated by a small distance $d\vec{\mathbf{s}}$, the following differential form is obtained:

$$dV = -\vec{\mathbf{E}}\cdot d\vec{\mathbf{s}} \tag{3.5.1}$$

In Cartesian coordinates, $\vec{\mathbf{E}} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}$ and $d\vec{\mathbf{s}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}},$ we have

$$dV = \left(E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}}\right)\cdot\left(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}\right) = E_x dx + E_y dy + E_z dz \tag{3.5.2}$$

which implies

$$\boxed{E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}} \tag{3.5.3}$$

By introducing a differential quantity called the "del (gradient) operator"

$$\nabla \equiv \frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}} \tag{3.5.4}$$

the electric field can be written as

$$\vec{\mathbf{E}} = E_x\hat{\mathbf{i}} + E_y\hat{\mathbf{j}} + E_z\hat{\mathbf{k}} = -\left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} + \frac{\partial V}{\partial z}\hat{\mathbf{k}}\right) = -\left(\frac{\partial}{\partial x}\hat{\mathbf{i}} + \frac{\partial}{\partial y}\hat{\mathbf{j}} + \frac{\partial}{\partial z}\hat{\mathbf{k}}\right)V = -\nabla V$$

$$\vec{\mathbf{E}} = -\nabla V \tag{3.5.5}$$

Notice that ∇ operates on a scalar quantity (electric potential) and results in a vector quantity (electric field). Mathematically, we can think of $\vec{\mathbf{E}}$ as the negative of the *gradient* of the electric potential V. Physically, the negative sign implies that if

V increases as a positive charge moves along some direction, say x, with $\partial V / \partial x > 0$, then there is a non-vanishing component of $\vec{\mathbf{E}}$ in the opposite direction $(-E_x \neq 0)$. In the case of gravity, if the gravitational potential increases when a mass is lifted a distance h, the gravitational force must be downward.

If the charge distribution possesses spherical symmetry, then the resulting electric field is a function of the radial distance r, i.e., $\vec{\mathbf{E}} = E_r \hat{\mathbf{r}}$. In this case, $dV = -E_r\, dr$. If $V(r)$ is known, then $\vec{\mathbf{E}}$ may be obtained as

$$\boxed{\vec{\mathbf{E}} = E_r \hat{\mathbf{r}} = -\left(\frac{dV}{dr}\right)\hat{\mathbf{r}}} \tag{3.5.6}$$

For example, the electric potential due to a point charge q is $V(r) = q / 4\pi\varepsilon_0 r$. Using the above formula, the electric field is simply $\vec{\mathbf{E}} = (q/4\pi\varepsilon_0 r^2)\hat{\mathbf{r}}$.

3.5.1 Gradient and Equipotentials

Suppose a system in two dimensions has an electric potential $V(x, y)$. The curves characterized by constant $V(x, y)$ are called equipotential curves. Examples of equipotential curves are depicted in Figure 3.5.1 below.

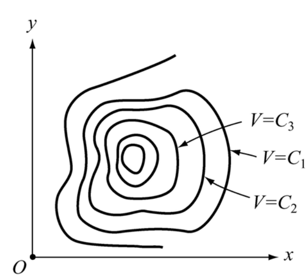


Figure 3.5.1 Equipotential curves

In three dimensions we have equipotential surfaces and they are described by $V(x, y, z)$ =constant. Since $\vec{\mathbf{E}} = -\nabla V$, we can show that the direction of $\vec{\mathbf{E}}$ is always perpendicular to the equipotential through the point. Below we give a proof in two dimensions. Generalization to three dimensions is straightforward.

Proof:

Referring to Figure 3.5.2, let the potential at a point $P(x, y)$ be $V(x, y)$. How much is V changed at a neighboring point $P(x + dx, y + dy)$? Let the difference be written as

$$dV = V(x+dx, y+dy) - V(x,y)$$
$$= \left[V(x,y) + \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy + \cdots \right] - V(x,y) \approx \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy \tag{3.5.7}$$

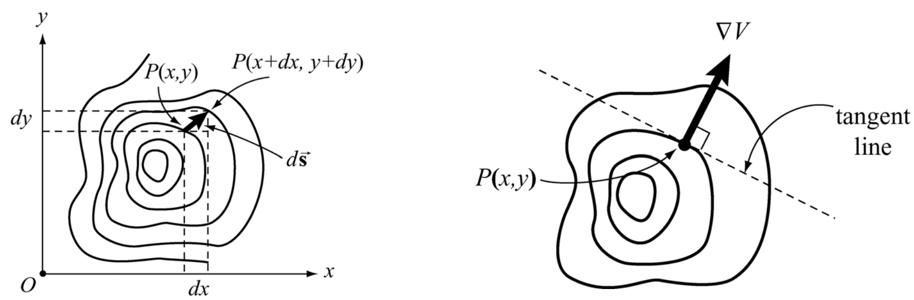


Figure 3.5.2 Change in V when moving from one equipotential curve to another

With the displacement vector given by $d\vec{\mathbf{s}} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$, we can rewrite dV as

$$dV = \left(\frac{\partial V}{\partial x}\hat{\mathbf{i}} + \frac{\partial V}{\partial y}\hat{\mathbf{j}} \right) \cdot \left(dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} \right) = (\nabla V) \cdot d\mathbf{s} = -\vec{\mathbf{E}} \cdot d\vec{\mathbf{s}} \tag{3.5.8}$$

If the displacement $d\vec{\mathbf{s}}$ is along the tangent to the equipotential curve through $P(x,y)$, then $dV = 0$ because V is constant everywhere on the curve. This implies that $\vec{\mathbf{E}} \perp d\vec{\mathbf{s}}$ along the equipotential curve. That is, $\vec{\mathbf{E}}$ is perpendicular to the equipotential. In Figure 3.5.3 we illustrate some examples of equipotential curves. In three dimensions they become equipotential surfaces. From Eq. (3.5.8), we also see that the change in potential dV attains a maximum when the gradient ∇V is parallel to $d\vec{\mathbf{s}}$:

$$\max\left(\frac{dV}{ds}\right) = \left|\nabla V\right| \tag{3.5.9}$$

Physically, this means that ∇V always points in the direction of maximum rate of change of V with respect to the displacement s.

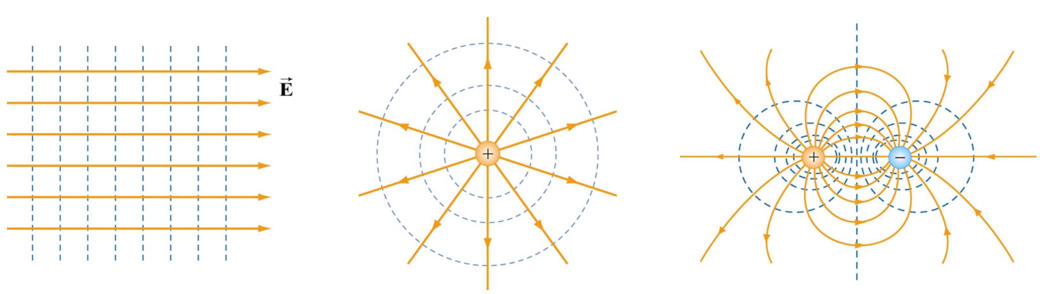


Figure 3.5.3 Equipotential curves and electric field lines for (a) a constant $\vec{\mathbf{E}}$ field, (b) a point charge, and (c) an electric dipole.

The properties of equipotential surfaces can be summarized as follows:

(i) The electric field lines are perpendicular to the equipotentials and point from higher to lower potentials.

(ii) By symmetry, the equipotential surfaces produced by a point charge form a family of concentric spheres, and for constant electric field, a family of planes perpendicular to the field lines.

(iii) The tangential component of the electric field along the equipotential surface is zero, otherwise non-vanishing work would be done to move a charge from one point on the surface to the other.

(iv) No work is required to move a particle along an equipotential surface.

A useful analogy for equipotential curves is a topographic map (Figure 3.5.4). Each contour line on the map represents a fixed elevation above sea level. Mathematically it is expressed as $z = f(x, y) = \text{constant}$. Since the gravitational potential near the surface of Earth is $V_g = gz$, these curves correspond to gravitational equipotentials.

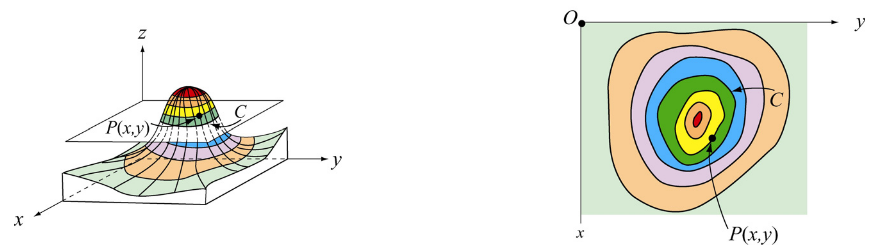


Figure 3.5.4 A topographic map

Example 3.1: Uniformly Charged Rod

Consider a non-conducting rod of length ℓ having a uniform charge density λ. Find the electric potential at P, a perpendicular distance y above the midpoint of the rod.

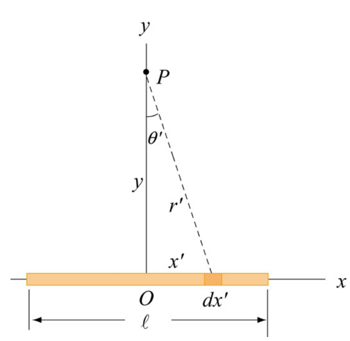


Figure 3.5.5 A non-conducting rod of length ℓ and uniform charge density λ.

Solution:

Consider a differential element of length dx' which carries a charge $dq = \lambda\, dx'$, as shown in Figure 3.5.5. The source element is located at $(x',0)$, while the field point P is located on the y-axis at $(0,y)$. The distance from dx' to P is $r = (x'^2 + y^2)^{1/2}$. Its contribution to the potential is given by

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda\, dx'}{(x'^2 + y^2)^{1/2}}$$

Taking V to be zero at infinity, the total potential due to the entire rod is

$$\begin{aligned} V &= \frac{\lambda}{4\pi\varepsilon_0}\int_{-\ell/2}^{\ell/2}\frac{dx'}{\sqrt{x'^2 + y^2}} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[x' + \sqrt{x'^2 + y^2}\right]\Bigg|_{-\ell/2}^{\ell/2} \\ &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}{-(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}\right] \end{aligned} \qquad (3.5.10)$$

where we have used the integration formula

$$\int \frac{dx'}{\sqrt{x'^2 + y^2}} = \ln\left(x' + \sqrt{x'^2 + y^2}\right)$$

A plot of $V(y)/V_0$, where $V_0 = \lambda / 4\pi\varepsilon_0$, as a function of y/ℓ is shown in Figure 3.5.6

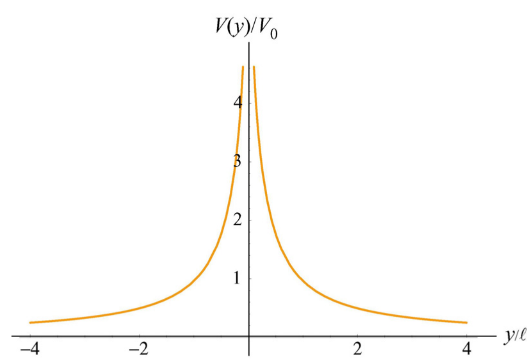


Figure 3.5.6 Electric potential along the axis that passes through the midpoint of a non-conducting rod.

In the limit $\ell \gg y$, the potential becomes

$$
\begin{aligned}
V &= \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(\ell/2)+\ell/2\sqrt{1+(2y/\ell)^2}}{-(\ell/2)+\ell/2\sqrt{1+(2y/\ell)^2}}\right] = \frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{1+\sqrt{1+(2y/\ell)^2}}{-1+\sqrt{1+(2y/\ell)^2}}\right] \\
&\approx \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{2}{2y^2/\ell^2}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{\ell^2}{y^2}\right) \\
&= \frac{\lambda}{2\pi\varepsilon_0}\ln\left(\frac{\ell}{y}\right)
\end{aligned}
\tag{3.5.11}
$$

The corresponding electric field can be obtained as

$$
\boxed{E_y = -\frac{\partial V}{\partial y} = \frac{\lambda}{2\pi\varepsilon_0 y}\frac{\ell/2}{\sqrt{(\ell/2)^2+y^2}}}
$$

in complete agreement with the result obtained in Eq. (2.10.9).

Example 3.2: Uniformly Charged Ring

Consider a uniformly charged ring of radius R and charge density λ (Figure 3.5.7). What is the electric potential at a distance z from the central axis?

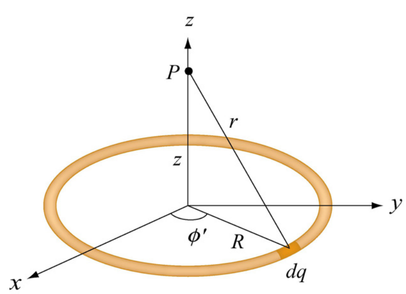


Figure 3.5.7 A non-conducting ring of radius R with uniform charge density λ.

Solution:

Consider a small differential element $d\ell = R\,d\phi'$ on the ring. The element carries a charge $dq = \lambda\,d\ell = \lambda R\,d\phi'$, and its contribution to the electric potential at P is

$$
dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R\,d\phi'}{\sqrt{R^2+z^2}}
$$

The electric potential at P due to the entire ring is

$$V = \int dV = \frac{1}{4\pi\varepsilon_0}\frac{\lambda R}{\sqrt{R^2+z^2}}\oint d\phi' = \frac{1}{4\pi\varepsilon_0}\frac{2\pi\lambda R}{\sqrt{R^2+z^2}} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}} \qquad (3.5.12)$$

where we have substituted $Q = 2\pi R\lambda$ for the total charge on the ring. In the limit $z \gg R$, the potential approaches its "point-charge" limit:

$$V \approx \frac{1}{4\pi\varepsilon_0}\frac{Q}{z}$$

From Eq. (3.5.12), the z-component of the electric field may be obtained as

$$E_z = -\frac{\partial V}{\partial z} = -\frac{\partial}{\partial z}\left(\frac{1}{4\pi\varepsilon_0}\frac{Q}{\sqrt{R^2+z^2}}\right) = \frac{1}{4\pi\varepsilon_0}\frac{Qz}{(R^2+z^2)^{3/2}} \qquad (3.5.13)$$

in agreement with Eq. (2.10.14).

Example 3.3: Uniformly Charged Disk

Consider a uniformly charged disk of radius R and charge density σ lying in the xy-plane. What is the electric potential at a distance z from the central axis?

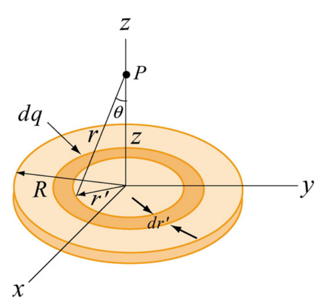


Figure 3.4.3 A non-conducting disk of radius R and uniform charge density σ.

Solution:

Consider a circular ring of radius r' and width dr'. The charge on the ring is $dq' = \sigma dA' = \sigma(2\pi r' dr')$. The field point P is located along the z-axis a distance z from the plane of the disk. From the figure, we also see that the distance from a point on the ring to P is $r = (r'^2 + z^2)^{1/2}$. Therefore, the contribution to the electric potential at P is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(2\pi r' dr')}{\sqrt{r'^2+z^2}}$$

By summing over all the rings that make up the disk, we have

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_0^R \frac{2\pi r' dr'}{\sqrt{r'^2+z^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{r'^2+z^2}\right]\Bigg|_0^R = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2}-|z|\right] \tag{3.5.14}$$

In the limit $|z| \gg R$,

$$\sqrt{R^2+z^2} = |z|\left(1+\frac{R^2}{z^2}\right)^{1/2} = |z|\left(1+\frac{R^2}{2z^2}+\cdots\right),$$

and the potential simplifies to the point-charge limit:

$$V \approx \frac{\sigma}{2\varepsilon_0}\cdot\frac{R^2}{2|z|} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma(\pi R^2)}{|z|} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{|z|}$$

As expected, at large distance, the potential due to a non-conducting charged disk is the same as that of a point charge *Q*. A comparison of the electric potentials of the disk and a point charge is shown in Figure 3.4.4.

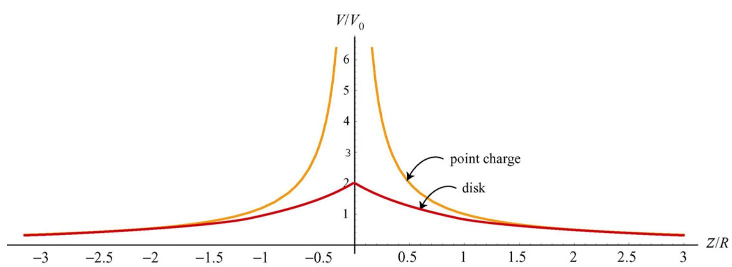


Figure 3.4.4 Comparison of the electric potentials of a non-conducting disk and a point charge. The electric potential is measured in terms of $V_0 = Q/4\pi\varepsilon_0 R$.

Note that the electric potential at the center of the disk ($z=0$) is finite, and its value is

$$V_c = \frac{\sigma R}{2\varepsilon_0} = \frac{Q}{\pi R^2}\cdot\frac{R}{2\varepsilon_0} = \frac{1}{4\pi\varepsilon_0}\frac{2Q}{R} = 2V_0 \tag{3.5.15}$$

This is the amount of work that needs to be done to bring a unit charge from infinity and place it at the center of the disk.

The corresponding electric field at *P* can be obtained as:

$$E_z = -\frac{\partial V}{\partial z} = \frac{\sigma}{2\varepsilon_0}\left[\frac{z}{|z|}-\frac{z}{\sqrt{R^2+z^2}}\right] \tag{3.5.16}$$

which agrees with Eq. (2.10.18). In the limit $R \gg z$, the above equation becomes $E_z = \sigma / 2\varepsilon_0$, which is the electric field for an infinitely large non-conducting sheet.

Example 3.4: Calculating Electric Field from Electric Potential

Suppose the electric potential due to a certain charge distribution can be written in Cartesian Coordinates as

$$V(x, y, z) = Ax^2y^2 + Bxyz$$

where A, B and C are constants. What is the associated electric field?

Solution:

The electric field can be found by using Eq. (3.5.3):

$$\begin{aligned} E_x &= -\frac{\partial V}{\partial x} = -2Axy^2 - Byz \\ E_y &= -\frac{\partial V}{\partial y} = -2Ax^2y - Bxz \\ E_z &= -\frac{\partial V}{\partial z} = -Bxy \end{aligned}$$

Therefore, the electric field is $\vec{\mathbf{E}} = (-2Axy^2 - Byz)\,\hat{\mathbf{i}} - (2Ax^2y + Bxz)\,\hat{\mathbf{j}} - Bxy\,\hat{\mathbf{k}}$.

3.6 Summary

- A force $\vec{\mathbf{F}}$ is **conservative** if the line integral of the force around a closed loop vanishes:

$$\oint \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{s}} = 0$$

- The change in potential energy associated with a conservative force $\vec{\mathbf{F}}$ acting on an object as it moves from A to B is

$$\Delta U = U_B - U_A = -\int_A^B \vec{\mathbf{F}} \cdot d\,\vec{\mathbf{s}}$$

- The **electric potential difference** ΔV between points *A* and *B* in an electric field $\vec{\mathbf{E}}$ is given by

$$\Delta V = V_B - V_A = \frac{\Delta U}{q_0} = -\int_A^B \vec{\mathbf{E}} \cdot d\vec{\mathbf{s}}$$

The quantity represents the amount of work done per unit charge to move a test charge q_0 from point *A* to *B*, without changing its kinetic energy.

- The electric potential due to a point charge Q at a distance *r* away from the charge is

$$V = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r}$$

For a collection of charges, using the superposition principle, the electric potential is

$$V = \frac{1}{4\pi\varepsilon_0}\sum_i \frac{Q_i}{r_i}$$

- The **potential energy** associated with two point charges q_1 and q_2 separated by a distance r_{12} is

$$U = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r_{12}}$$

- From the electric potential V, the electric field may be obtained by taking the **gradient** of V:

$$\vec{\mathbf{E}} = -\nabla V$$

In Cartesian coordinates, the components may be written as

$$E_x = -\frac{\partial V}{\partial x}, \quad E_y = -\frac{\partial V}{\partial y}, \quad E_z = -\frac{\partial V}{\partial z}$$

- The electric potential due to a continuous charge distribution is

$$V = \frac{1}{4\pi\varepsilon_0}\int \frac{dq}{r}$$

3.7 Problem-Solving Strategy: Calculating Electric Potential

In this chapter, we showed how electric potential can be calculated for both the discrete and continuous charge distributions. Unlike electric field, electric potential is a scalar quantity. For the discrete distribution, we apply the superposition principle and sum over individual contributions:

$$V = k_e \sum_i \frac{q_i}{r_i}$$

For the continuous distribution, we must evaluate the integral

$$V = k_e \int \frac{dq}{r}$$

In analogy to the case of computing the electric field, we use the following steps to complete the integration:

(1) Start with $dV = k_e \dfrac{dq}{r}$.

(2) Rewrite the charge element dq as

$$dq = \begin{cases} \lambda\, dl & \text{(length)} \\ \sigma\, dA & \text{(area)} \\ \rho\, dV & \text{(volume)} \end{cases}$$

depending on whether the charge is distributed over a length, an area, or a volume.

(3) Substitute dq into the expression for dV.

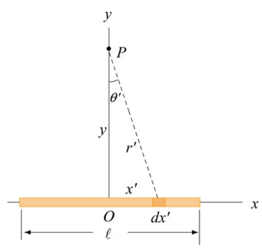
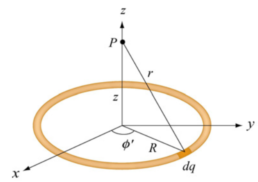
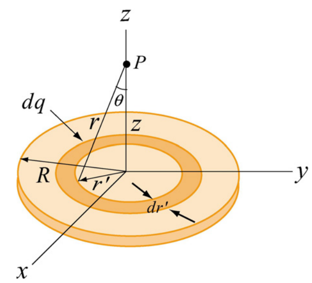
(4) Specify an appropriate coordinate system and express the differential element (dl, dA or dV) and r in terms of the coordinates (see Table 2.1.)

(5) Rewrite dV in terms of the integration variable.

(6) Complete the integration to obtain V.

Using the result obtained for V, one may calculate the electric field by $\vec{\mathbf{E}} = -\nabla V$. Furthermore, the accuracy of the result can be readily checked by choosing a point P which lies sufficiently far away from the charge distribution. In this limit, if the charge distribution is of finite extent, the field should behave as if the distribution were a point charge, and falls off as $1/r^2$.

Below we illustrate how the above methodologies can be employed to compute the electric potential for a line of charge, a ring of charge and a uniformly charged disk.

	Charged Rod	**Charged Ring**	**Charged disk**
Figure			
(2) Express dq in terms of charge density	$dq = \lambda\, dx'$	$dq = \lambda\, dl$	$dq = \sigma\, dA$
(3) Substitute dq into expression for dV	$dV = k_e \dfrac{\lambda\, dx'}{r}$	$dV = k_e \dfrac{\lambda\, dl}{r}$	$dV = k_e \dfrac{\sigma dA}{r}$
(4) Rewrite r and the differential element in terms of the appropriate coordinates	dx' $r = \sqrt{x'^2 + y^2}$	$dl = R\, d\phi'$ $r = \sqrt{R^2 + z^2}$	$dA = 2\pi r'\, dr'$ $r = \sqrt{r'^2 + z^2}$
(5) Rewrite dV	$dV = k_e \dfrac{\lambda\, dx'}{(x'^2 + y^2)^{1/2}}$	$dV = k_e \dfrac{\lambda R\, d\phi'}{(R^2 + z^2)^{1/2}}$	$dV = k_e \dfrac{2\pi\sigma r'\, dr'}{(r'^2 + z^2)^{1/2}}$
(6) Integrate to get V	$V = \dfrac{\lambda}{4\pi\varepsilon_0} \int_{-\ell/2}^{\ell/2} \dfrac{dx'}{\sqrt{x'^2 + y^2}}$ $= \dfrac{\lambda}{4\pi\varepsilon_0} \ln\left[\dfrac{(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}{-(\ell/2) + \sqrt{(\ell/2)^2 + y^2}}\right]$	$V = k_e \dfrac{R\lambda}{(R^2 + z^2)^{1/2}} \oint d\phi'$ $= k_e \dfrac{(2\pi R\lambda)}{\sqrt{R^2 + z^2}}$ $= k_e \dfrac{Q}{\sqrt{R^2 + z^2}}$	$V = k_e 2\pi\sigma \int_0^R \dfrac{r'\, dr'}{(r'^2 + z^2)^{1/2}}$ $= 2k_e \pi\sigma\left(\sqrt{z^2 + R^2} - \|z\|\right)$ $= \dfrac{2k_e Q}{R^2}\left(\sqrt{z^2 + R^2} - \|z\|\right)$
Derive E from V	$E_y = -\dfrac{\partial V}{\partial y}$ $= \dfrac{\lambda}{2\pi\varepsilon_0 y} \dfrac{\ell/2}{\sqrt{(\ell/2)^2 + y^2}}$	$E_z = -\dfrac{\partial V}{\partial z} = \dfrac{k_e Q z}{(R^2 + z^2)^{3/2}}$	$E_z = -\dfrac{\partial V}{\partial z} = \dfrac{2k_e Q}{R^2}\left(\dfrac{z}{\|z\|} - \dfrac{z}{\sqrt{z^2 + R^2}}\right)$
Point-charge limit for E	$E_y \approx \dfrac{k_e Q}{y^2} \quad y \gg \ell$	$E_z \approx \dfrac{k_e Q}{z^2} \quad z \gg R$	$E_z \approx \dfrac{k_e Q}{z^2} \quad z \gg R$

3.8 Solved Problems

3.8.1 Electric Potential Due to a System of Two Charges

Consider a system of two charges shown in Figure 3.8.1.

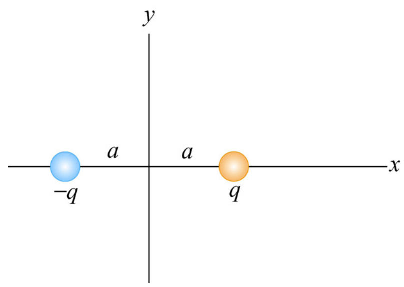


Figure 3.8.1 Electric dipole

Find the electric potential at an arbitrary point on the *x* axis and make a plot.

Solution:

The electric potential can be found by the superposition principle. At a point on the *x* axis, we have

$$V(x) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|x-a|} + \frac{1}{4\pi\varepsilon_0}\frac{(-q)}{|x+a|} = \frac{q}{4\pi\varepsilon_0}\left[\frac{1}{|x-a|} - \frac{1}{|x+a|}\right]$$

The above expression may be rewritten as

$$\frac{V(x)}{V_0} = \frac{1}{|x/a-1|} - \frac{1}{|x/a+1|}$$

where $V_0 = q/4\pi\varepsilon_0 a$. The plot of the dimensionless electric potential as a function of x/a. is depicted in Figure 3.8.2.

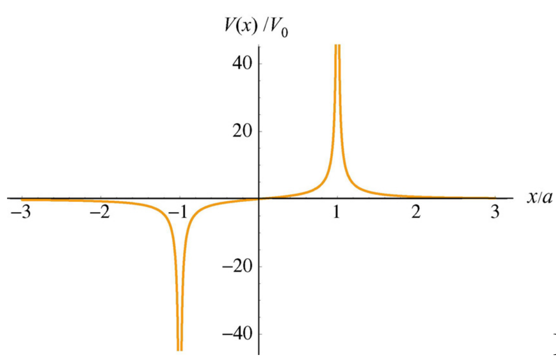


Figure 3.8.2

As can be seen from the graph, $V(x)$ diverges at $x/a = \pm 1$, where the charges are located.

3.8.2 Electric Dipole Potential

Consider an electric dipole along the *y*-axis, as shown in the Figure 3.8.3. Find the electric potential V at a point P in the *x*-*y* plane, and use V to derive the corresponding electric field.

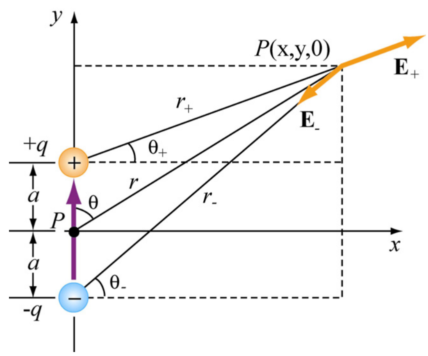


Figure 3.8.3

By superposition principle, the potential at P is given by

$$V = \sum_i V_i = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

where $r_\pm^{\ 2} = r^2 + a^2 \mp 2ra\cos\theta$. If we take the limit where $r \gg a$, then

$$\frac{1}{r_\pm} = \frac{1}{r}\left[1 + (a/r)^2 \mp 2(a/r)\cos\theta\right]^{-1/2} = \frac{1}{r}\left[1 - \frac{1}{2}(a/r)^2 \pm (a/r)\cos\theta + \cdots\right]$$

and the dipole potential can be approximated as

$$\begin{aligned} V &= \frac{q}{4\pi\varepsilon_0 r}\left[1 - \frac{1}{2}(a/r)^2 + (a/r)\cos\theta - 1 + \frac{1}{2}(a/r)^2 + (a/r)\cos\theta + \cdots\right] \\ &\approx \frac{q}{4\pi\varepsilon_0 r}\cdot\frac{2a\cos\theta}{r} = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\vec{\mathbf{p}}\cdot\hat{\mathbf{r}}}{4\pi\varepsilon_0 r^2} \end{aligned}$$

where $\vec{\mathbf{p}} = 2aq\,\hat{\mathbf{j}}$ is the electric dipole moment. In spherical polar coordinates, the gradient operator is

$$\vec{\nabla} = \frac{\partial}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\hat{\boldsymbol{\varphi}}$$

Since the potential is now a function of both r and θ, the electric field will have components along the $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ directions. Using $\vec{\mathbf{E}} = -\nabla V$, we have

$$\boxed{E_r = -\frac{\partial V}{\partial r} = \frac{p\cos\theta}{2\pi\varepsilon_0 r^3}, \quad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}, \quad E_\phi = 0}$$

3.8.3 Electric Potential of an Annulus

Consider an annulus of uniform charge density σ, as shown in Figure 3.8.4. Find the electric potential at a point P along the symmetric axis.

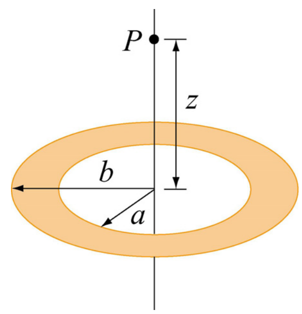


Figure 3.8.4 An annulus of uniform charge density.

Solution:

Consider a small differential element dA at a distance r away from point P. The amount of charge contained in dA is given by

$$dq = \sigma dA = \sigma (r' d\theta) dr'$$

Its contribution to the electric potential at P is

$$dV = \frac{1}{4\pi\varepsilon_0}\frac{dq}{r} = \frac{1}{4\pi\varepsilon_0}\frac{\sigma r' dr' d\theta}{\sqrt{r'^2 + z^2}}$$

Integrating over the entire annulus, we obtain

$$V = \frac{\sigma}{4\pi\varepsilon_0}\int_a^b \int_0^{2\pi} \frac{r' dr' d\theta}{\sqrt{r'^2 + z^2}} = \frac{2\pi\sigma}{4\pi\varepsilon_0}\int_a^b \frac{r' ds}{\sqrt{r'^2 + z^2}} = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{b^2 + z^2} - \sqrt{a^2 + z^2}\right]$$

where we have made used of the integral

$$\int \frac{ds\, s}{\sqrt{s^2+z^2}} = \sqrt{s^2+z^2}$$

Notice that in the limit $a \to 0$ and $b \to R$, the potential becomes

$$V = \frac{\sigma}{2\varepsilon_0}\left[\sqrt{R^2+z^2} - |z|\right]$$

which coincides with the result of a non-conducting disk of radius R shown in Eq. (3.5.14).

3.8.4 Charge Moving Near a Charged Wire

A thin rod extends along the z-axis from $z=-d$ to $z=d$. The rod carries a positive charge Q uniformly distributed along its length $2d$ with charge density $\lambda = Q/2d$.

(a) Calculate the electric potential at a point $z > d$ along the z-axis.

(b) What is the change in potential energy if an electron moves from $z=4d$ to $z=3d$?

(c) If the electron started out at rest at the point $z=4d$, what is its velocity at $z=3d$?

Solutions:

(a) For simplicity, let's set the potential to be zero at infinity, $V(\infty)=0$. Consider an infinitesimal charge element $dq = \lambda\, dz'$ located at a distance z' along the z-axis. Its contribution to the electric potential at a point $z > d$ is

$$dV = \frac{\lambda}{4\pi\varepsilon_0}\frac{dz'}{z-z'}$$

Integrating over the entire length of the rod, we obtain

$$V(z) = \frac{\lambda}{4\pi\varepsilon_0}\int_{z+d}^{z-d}\frac{dz'}{z-z'} = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{z+d}{z-d}\right)$$

(b) Using the result derived in (a), the electrical potential at $z=4d$ is

$$V(z=4d) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{4d+d}{4d-d}\right) = \frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{5}{3}\right)$$

Similarly, the electrical potential at $z = 3d$ is

$$V(z=3d)=\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{3d+d}{3d-d}\right)=\frac{\lambda}{4\pi\varepsilon_0}\ln 2$$

The electric potential difference between the two points is

$$\Delta V=V(z=3d)-V(z=4d)=\frac{\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right)>0$$

Using the fact that the electric potential difference ΔV is equal to the change in potential energy per unit charge, we have

$$\Delta U=q\Delta V=-\frac{|e|\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right)<0$$

where $q=-|e|$ is the charge of the electron.

(c) If the electron starts out at rest at $z=4d$ then the change in kinetic energy is

$$\Delta K=\frac{1}{2}mv_f^{\,2}$$

By conservation of energy, the change in kinetic energy is

$$\Delta K=-\Delta U=\frac{|e|\lambda}{4\pi\varepsilon_0}\ln\left(\frac{6}{5}\right)>0$$

Thus, the magnitude of the velocity at $z=3d$ is

$$v_f=\sqrt{\frac{2|e|\lambda}{4\pi\varepsilon_0 m}\ln\left(\frac{6}{5}\right)}$$

3.9 Conceptual Questions

1. What is the difference between electric potential and electric potential energy?

2. A uniform electric field is parallel to the *x*-axis. In what direction can a charge be displaced in this field without any external work being done on the charge?

3. Is it safe to stay in an automobile with a metal body during severe thunderstorm? Explain.

4. Why are equipotential surfaces always perpendicular to electric field lines?

5. The electric field inside a hollow, uniformly charged sphere is zero. Does this imply that the potential is zero inside the sphere?

3.10 Additional Problems

3.10.1 Cube

How much work is done to assemble eight identical point charges, each of magnitude q, at the corners of a cube of side a?

3.10.2 Three Charges

Three charges with $q = 3.00\times10^{-18}$ C and $q_1 = 6\times10^{-6}$ C are placed on the x-axis, as shown in the figure 3.10.1. The distance between q and q_1 is a = 0.600 m.

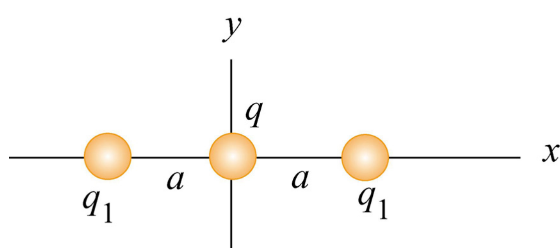


Figure 3.10.1

(a) What is the net force exerted on q by the other two charges q_1?

(b) What is the electric field at the origin due to the two charges q_1?

(c) What is the electric potential at the origin due to the two charges q_1?

3.10.3 Work Done on Charges

Two charges $q_1 = 3.0\,\mu\text{C}$ and $q_2 = -4.0\,\mu\text{C}$ initially are separated by a distance $r_0 = 2.0\,\text{cm}$. An external agent moves the charges until they are $r_f = 5.0\,\text{cm}$ apart.

(a) How much work is done by the *electric field* in moving the charges from r_0 to r_f? Is the work positive or negative?

(b) How much work is done by the e*xternal agent* in moving the charges from r_0 to r_f? Is the work positive or negative?

(c) What is the potential energy of the initial state where the charges are $r_0 = 2.0\,\text{cm}$ apart?

(d) What is the potential energy of the final state where the charges are $r_f = 5.0\,\text{cm}$ apart?

(e) What is the change in potential energy from the initial state to the final state?

3.10.4 Calculating *E* from *V*

Suppose in some region of space the electric potential is given by

$$V(x,y,z) = V_0 - E_0 z + \frac{E_0 a^3 z}{(x^2 + y^2 + z^2)^{3/2}}$$

where a is a constant with dimensions of length. Find the *x, y,* and the *z*-components of the associated electric field.

3.10.5 Electric Potential of a Rod

A rod of length L lies along the x-axis with its left end at the origin and has a non-uniform charge density $\lambda = \alpha x$,where α is a positive constant.

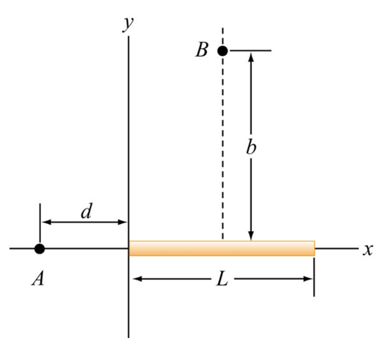


Figure 3.10.2

(a) What are the dimensions of α ?

(b) Calculate the electric potential at *A*.

(c) Calculate the electric potential at point *B* that lies along the perpendicular bisector of the rod a distance *b* above the *x*-axis.

3.10.6 Electric Potential

Suppose that the electric potential in some region of space is given by

$$V(x, y, z) = V_0 \exp(-k\,|\,z\,|) \cos kx\,.$$

Find the electric field everywhere. Sketch the electric field lines in the $x - z$ plane.

3.10.7 Calculating Electric Field from the Electric Potential

Suppose that the electric potential varies along the *x*-axis as shown in Figure 3.10.3 below.

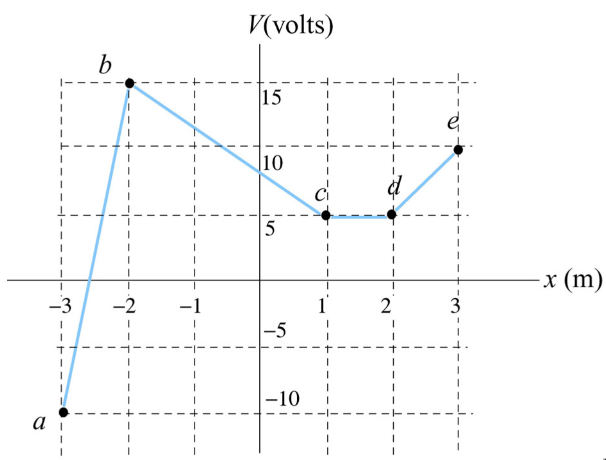


Figure 3.10.3

The potential does not vary in the *y*- or *z* -direction. Of the intervals shown (ignore the behavior at the end points of the intervals), determine the intervals in which E_x has

(a) its greatest absolute value. [Ans: 25 V/m in interval *ab*.]

(b) its least. [Ans: (b) 0 V/m in interval *cd*.]

(c) Plot E_x as a function of *x*.

(d) What sort of charge distributions would produce these kinds of changes in the potential? Where are they located? [Ans: sheets of charge extending in the *yz* direction located at points *b*, *c*, *d*, etc. along the *x*-axis. Note again that a sheet of charge with charge per unit area σ will *always* produce a jump in the normal component of the electric field of magnitude σ / ε_0].

3.10.8 Electric Potential and Electric Potential Energy

A right isosceles triangle of side a has charges q, $+2q$ and $-q$ arranged on its vertices, as shown in Figure 3.10.4.

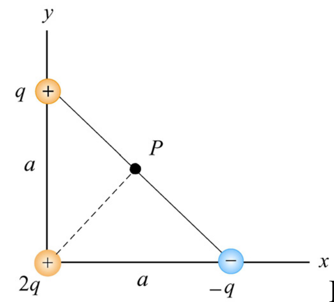


Figure 3.10.4

(a) What is the electric potential at point P, midway between the line connecting the $+q$ and $-q$ charges, assuming that $V = 0$ at infinity? [Ans: $q/\sqrt{2}\,\pi\varepsilon_0 a$.]

(b) What is the potential energy U of this configuration of three charges? What is the significance of the sign of your answer? [Ans: $-q^2/4\sqrt{2}\,\pi\varepsilon_0 a$, the negative sign means that work was done on the agent who assembled these charges in moving them in from infinity.]

(c) A fourth charge with charge $+3q$ is slowly moved in from infinity to point P. How much work must be done in this process? What is the significance of the sign of your answer? [Ans: $+3q^2/\sqrt{2}\,\pi\varepsilon_0 a$, the positive sign means that work was done by the agent who moved this charge in from infinity.]

3.10.9. Electric Field, Potential and Energy

Three charges, $+5Q$, $-5Q$, and $+3Q$ are located on the y-axis at $y = +4a$, $y = 0$, and $y = -4a$, respectively. The point P is on the x-axis at $x = 3a$.

(a) How much energy did it take to assemble these charges?

(b) What are the x, y, and z components of the electric field $\vec{\mathbf{E}}$ at P?

(c) What is the electric potential V at point P, taking $V = 0$ at infinity?

(d) A fourth charge of $+Q$ is brought to P from infinity. What are the x, y, and z components of the force $\vec{\mathbf{F}}$ that is exerted on it by the other three charges?

(e) How much work was done (by the external agent) in moving the fourth charge $+Q$ from infinity to P?

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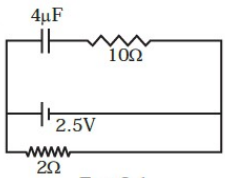
Physics home assignment 2

Chapter Two : Electrostatic Potential and Capacitance

1. A positively charged particle is released from rest in an uniform electric field. The electric potential energy of the charge
   (a) remains a constant because the electric field is uniform.
   (b) increases because the charge moves along the electric field.
   (c) decreases because the charge moves along the electric field.
   (d) decreases because the charge moves opposite to the electric field.
2. Two objects A and B are charged with equal charge The potential of A relative to B will be -
   (1) more (2) equal (3) less (4) indefinite
3. The potential due to a point charge at distance r is -
   (1) proportional to r (2) inversely proportional to r
   (3) proportional to r^2 (4) inversely proportional to r^2
4. The dimensions of potential difference are
   (1) $ML^2T^{-2}Q^{-1}$ (2) $MLT^{-2}Q^{-1}$ (3) $MT^{-2}Q^{-2}$ (4) $ML^2T^{-1}Q^{-1}$
5. An object is charged with positive charge. The potential at that object will be
   (1) positive only (2) negative only (3) zero always (4) may be positive, negative or zero.

6. The potential at 0.5 Å from a proton is
   (1) 0.5 volt (2) 8μ volt (3) 28.8 volt (4) 2 volt
7. Two metallic spheres which have equal charges, but their radii are different, are made to touch each other and then separated apart. The potential on the spheres will be -
   (1) same as before (2) more for bigger (3) more for smaller (4) equal
8. A conducting shell of radius 10 cm is charged with 3.2×10^{-1} C. The electric potential at a distance 4cm from its centre in volt be
   (1) 9×10^{-9} (2) 288 (3) 2.88×10^{-8} (4) zero

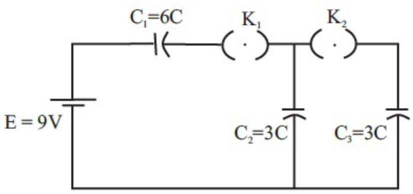
9. A capacitor of 4 μ F is connected as shown in the circuit.



The internal resistance of the battery is 0.5 Ω . The maximum amount of charge on the capacitor plates will be
(a) 0 (b) 4 μ C (c) 16 μ C (d) 8 μ C

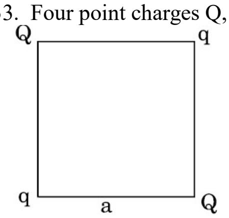
10. Equipotentials at a great distance from a collection of charges whose total sum is not zero are approximately
   (a) spheres. (b) planes. (c) paraboloids (d) ellipsoids.

11. Calculate the potential at a point P due to a charge of 4×10^{-7}C located 9 cm away.Hence obtain the work done in bringing a charge of 2×10^{-9} C from infinity to the point P. Does the answer depend on the path along which the charge is brought?
12. Calculate potential on the axis of a ring due to charge Q uniformly distributed along the ring of radius *R*.
13. Two metal spheres, one of radius *R* and the other of radius 2*R*, both have same surface charge density σ . They are brought in contact and separated. What will be new surface charge densities on them?
14. In the circuit shown in Figure, initially K1 is closed and K2 is open. What are the charges on each capacitors. Then K1 was opened and K2 was closed (order is important), What will be the charge on each capacitor now? [C = 1μF]

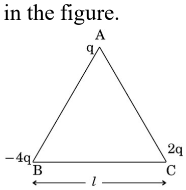


15. Two charges 5×10^{-8} C and -3×10^{-8} C are located 16 cm apart. At what point(s) on the line joining the two charges is the electric potential zero? Take the potential at infinity to be zero.
16. A regular hexagon of side 10 cm has a charge 5 μC at each of its vertices. Calculate the potential at the centre of the hexagon.
17. Two charges 2 μC and –2 μC are placed at points A and B 6 cm apart.
(a) Identify an equipotential surface of the system.

(b) What is the direction of the electric field at every point on this surface?
18. A parallel plate capacitor with air between the plates has a capacitance of 8 pF (1pF = 10^{-12} F). What will be the capacitance if the distance between the plates is reduced by half, and the space between them is filled with a substance of dielectric constant 6?
19. Three capacitors each of capacitance 9 pF are connected in series.
(a) What is the total capacitance of the combination?
(b) What is the potential difference across each capacitor if the combination is connected to a 120 V supply?
20. Three capacitors of capacitances 2 pF, 3 pF and 4 pF are connected in parallel.
(a) What is the total capacitance of the combination?
(b) Determine the charge on each capacitor if the combination is connected to a 100 V supply.
21. In a parallel plate capacitor with air between the plates, each plate has an area of 6×10^{-3} m^2 and the distance between the plates is 3 mm. Calculate the capacitance of the capacitor. If this capacitor is connected to a 100 V supply, what is the charge on each plate of the capacitor?
22. A 12pF capacitor is connected to a 50V battery. How much electrostatic energy is stored in the capacitor?
23. A 600pF capacitor is charged by a 200V supply. It is then disconnected from the supply and is connected to another uncharged 600 pF capacitor. How much electrostatic energy is lost in the process?
24. A charge of 8 mC is located at the origin. Calculate the work done in taking a small charge of –2 × 10–9 C from a point P (0, 0, 3 cm) to a point Q (0, 4 cm, 0), via a point R (0, 6 cm, 9 cm).
25. A cube of side *b* has a charge *q* at each of its vertices. Determine the potential and electric field due to this charge array at the centre of the cube.
26. Two tiny spheres carrying charges 1.5 μC and 2.5 μC are located 30 cm apart. Find the potential and electric field:
(a) at the mid-point of the line joining the two charges, and
(b) at a point 10 cm from this midpoint in a plane normal to the line and passing through the mid-point.
27. A spherical conducting shell of inner radius *r*1 and outer radius *r*2 has a charge *Q*. A charge *q* is placed at the centre of the shell. What is the surface charge density on the inner and outer surfaces of the shell?
28. Derive an expression for the electric potential at any point along the axial line of an electric dipole.
29. Draw the equipotential surfaces corresponding to a uniform electric field in the z-direction.
30. Derive an expression for the electric potential at any point along the axial line of an electric dipole.
31. Describe briefly the process of transferring the charge between the two plates of a parallel plate capacitor when connected to a battery. Derive an expression for the energy stored in a capacitor.
32. A parallel plate capacitor is charged by a battery to a potential difference V. It is disconnected from battery and then connected to another uncharged capacitor of the same capacitance. Calculate the ratio of the energy stored in the combination to the initial energy on the single capacitor.
33. Four point charges Q, q, Q and q are placed at the corners of a square of side ‘a’ as shown in the figure



. Find the potential energy of this system.
34. Three point charges q, – 4q and 2q are placed at the vertices of an equilateral triangle ABC of side *‘l’* as shown in the figure.



Find out the amount of the work done to separate the charges at infinite distance.

35. An electron is accelerated through a potential difference V. Write the expression for its final speed, if it was initially at rest
36. Two point charges q and –q are located at points (0, 0, –a) and (0, 0, a) respectively.
(a) Find the electrostatic potential at (0, 0, z) and (x, y, 0)
(b) How much work is done in moving a small test charge from the point (5, 0, 0) to (–7, 0, 0) along the x-axis ?
37. A capacitor of capacitance C1 is charged to a potential V1 while another capacitor of capacitance C2 is charged to a potential difference V2. The capacitors are now disconnected from their respective charging batteries and connected in parallel to each other.
(a) Find the total energy stored in the two capacitors before they are connected.
(b) Find the total energy stored in the parallel combination of the two capacitors.
(c) Explain the reason for the difference of energy in parallel combination in comparison to the total energy before they are connected.
38. Define the capacitance of a capacitor. Obtain the expression for the capacitance of a parallel plate capacitor in vacuum in terms of plate area A and separation d between the plates.

(b) A slab of material of dielectric constant K has the same area as the plates of a parallel plate capacitor but has a thickness 3d/4. Find the ratio of the capacitance with dielectric inside it to its capacitance without the dielectric.

39. Write two properties by which electric potential is related to the electric field.

40. In the following arrangement of capacitors, the energy stored in the 6 μF capacitor is E. Find the value of the following :

(i) Energy stored in 12 μF capacitor.

(ii) Energy stored in 3 μF capacitor.

(iii) Total energy drawn from the battery.